A curious arithmetic of fractal dimension for polyadic Cantor sets

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Abstract

Fractal sets, by definition, are non-differentiable, however their dimension can be continuous, differentiable, and arithmetically manipulable as function of their construction parameters. A new arithmetic for fractal dimension of polyadic Cantor sets is introduced by means of properly defining operators for the addition, subtraction, multiplication, and division. The new operators have the usual properties of the corresponding operations with real numbers. The combination of an infinitesimal change of fractal dimension with these arithmetic operators allows the manipulation of fractal dimension with the tools of calculus.

Key words: Fractal geometry, Fractal dimension, Arithmetical operators, Polyadic Cantor sets

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1 Introduction

This paper is motivated by the recent introduction of a series of arithmetic operators for fractal dimension of hyperhelices [2,3] by Carlos D. Toledo-Suárez [1]. Let us briefly recall that a hyperhelix of order N is defined to be a self-similar object consisting of a thin elastic rod wound into a helix, which is itself wound into a larger helix, until this process has been repeated N times. A hyperhelix fractal results when N tends to infinity. The Hausdorff-Besicovitch dimension of a hyperhelix fractal can be any desired number above unity. However, the interpretation of hyperhelices with fractal dimension larger than three, which unavoidably intersect themselves, is not obvious. In fact, this author is attempted to point out that geometrical considerations forbids hyperhelix fractal dimensions larger than three. This is the main inconvenient with the fractal dimension arithmetics introduced by Toledo-Suárez [1]. Could such drawbacks be circumvented by using another kind of fractal?

This short paper introduces an algebraic structure for the fractal dimension of polyadic Cantor fractals, resulting from iterative replacement of segments in the unit interval by scaled versions of them, whose Hausdorff-Besicovitch dimension can be any real number in the interval $[0, 1]$. The new arithmetic preserves this property avoiding the appearance of non-geometrically feasible fractal dimensions larger than unity. The contents of this paper are as follows. The next section recalls the characteristics of symmetrical, polyadic, Cantor fractal sets. Section 3, presents the new arithmetical operators for the addition, subtraction, product, and division of the fractal dimension, summarizing their main properties. Finally, the last section is devoted to the main conclusions.

2 Polyadic Cantor fractals

Polyadic, or generalized, Cantor sets are defined as follows [4,5]. The first step ($S = 0$) is to take a segment of unit length, referred to as the initiator. In the next step, $S = 1$, the segment is replaced by N non-overlapping copies of the initiator, each one scaled by a factor $\gamma < 1$. For N odd, as shown in Fig. 1(a), one copy lies exactly centered in the interval, and the remaining ones are distributed such that $\lfloor N/2 \rfloor$ copies are placed completely to the left of the interval and the remaining $\lfloor N/2 \rfloor$ copies are placed wholly to its right, where $\lfloor N/2 \rfloor$ is the greatest integer less than or equal to $N/2$; additionally, each copy among these $N-1$ ones is separated by a fixed distance, let us say ε. For N even, as shown in Fig. 1(b), one half of the copies is placed completely to the left of the interval and the other half completely to its right, with each copy separated by ε. At the following construction stages of the the fractal set, $S = 2, 3, \ldots$, the generation process is repeated over and over again for each segment in the previous stage. Strictly speaking, the Cantor set is the limit of this procedure as $S \to \infty$, which is composed of geometric points distributed so that each point lies arbitrarily close of other points of the set, being the S-th stage Cantor set usually referred to as a pre-fractal or physical fractal.

Symmetrical polyadic Cantor fractals are characterized by three parameters, the number of self-similar copies N, the scaling factor γ, and the width of the outermost gap at the first stage, ε, here on referred to as the lacunarity parameter, as in most of the previous technical papers dealing with polyadic Cantor fractals in engineering applications [6,7,8]. The similarity dimension of all polyadic Cantor fractals is $D = \ln(N)/\ln(\gamma^{-1}) = -\ln(N)/\ln(\gamma)$, which is independent of the lacunarity parameter.

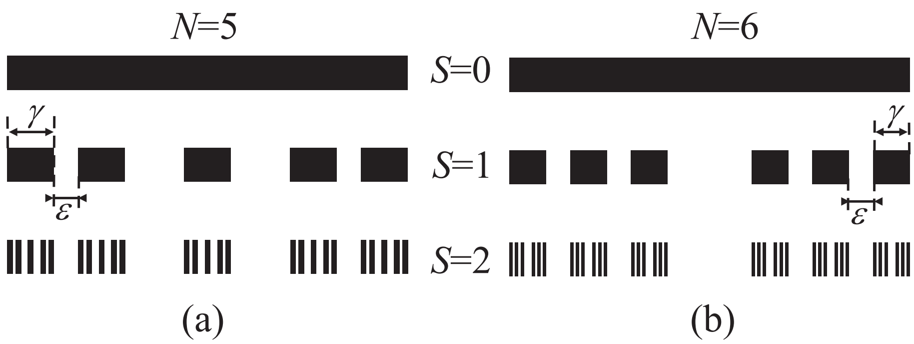

Fig. 1. First two stages ($S = 1$ and $S = 2$) of polyadic Cantor fractal sets with $N = 5$ (a) and $N = 6$ (b), showing the definition of both the scale factor (γ) and the lacunarity parameter (ε).

The three parameters of a polyadic Cantor set must satisfy certain constraints in order to avoid overlapping between the copies of the initiator. Let us recall such almost obvious results for the sake of completeness. On the one hand, the maximum value of the scaling factor depends on the value of N, such that $0 < \gamma < \gamma_{\max} = 1/N$. On the other hand, for each N and γ, there are two extreme values for ε. The first one is $\varepsilon_{\min} = 0$, for which the highest lacunar fractal is obtained, i.e., that with the largest possible gap. For N even, the central gap has a width equal to $1 - N\gamma$, and, for N odd, both large gaps surrounding the central well have a width of $(1 - N\gamma)/2$. The other extreme value is

$$\varepsilon_{\max} = \begin{cases} \dfrac{1 - N\gamma}{N - 2}, & \text{even } N, \\[2em] \dfrac{1 - N\gamma}{N - 3}, & \text{odd } N, \end{cases}$$

where for even (odd) N two (three) wells are joined together in the center, without any gap in the central region. The width of the $N - 2$ gaps in this case is equal to $\varepsilon_{\max}$. Thus, the corresponding lacunarity is lower than that for $\varepsilon = 0$, but not the smallest one, which is obtained for the most regular distribution, where the gaps and wells have the same width at the first stage ($S = 1$) given by

$$\varepsilon_{\rm reg} = \frac{1 - N\,\gamma}{N - 1}.$$

Note that $0 < \varepsilon_{\rm reg} < \varepsilon_{\rm max}$.

3 Arithmetic of fractal dimension

Let us take two N-adic Cantor fractals A and B with scaling factors (fractal dimensions) γ_A (D_A) and γ_B (D_B) , respectively. Their fractal dimensions can be combined by means of the following four operators in order to obtain a new N-adic Cantor fractal C with scaling factor γ_C and fractal dimension D_C. Note that the independency of the similarity dimension on the lacunarity parameter avoids the need of its incorporation into the dimension arithmetic which follows.

3.1 Addition ($\oplus$)

Let us define the N-adic Cantor fractal $C = A \oplus B$ by means of $\gamma_C = \gamma_A\,\gamma_B$, resulting in

$$\frac{1}{D_C} = \frac{1}{D_A} + \frac{1}{D_B}, \qquad D_C = D_A \oplus D_B = \frac{D_A\,D_B}{D_A + D_B},$$

where, by abuse of notation, the operator $\oplus$ has also been applied to the fractal dimensions of the fractals A and B. The present definition of the operator $\oplus$ is consistent, since for $0 < \gamma_A, \gamma_B < 1/N$, then $0 < \gamma_C < 1/N$, i.e., for $0 < D_A, D_B < 1$, then $0 < D_C < 1$.

The operator $\oplus$ is commutative ($D_A \oplus D_B = D_B \oplus D_A$) and associative

$$D_A \oplus (D_B \oplus D_C) = \frac{1}{\frac{1}{D_A} + \frac{1}{D_B} + \frac{1}{D_C}} = (D_A \oplus D_B) \oplus D_C.$$

The void set, i.e., the N-adic Cantor fractal Z with $\gamma_Z = 0$ and $D_Z = 0$, is the absorbing element of $\oplus$, since $D_Z = D_A \oplus D_Z$ for all A.

There is no identity element Y for $\oplus$, such that $D_A = D_A \oplus D_Y$, since the only possibility is to take $\gamma_Y = 1$ and $D_Y = +\infty$, which it is not a geometrically valid N-adic Cantor fractal.

Let U be the unit segment, with $D_U = 1$ and $\gamma_U = 1/N$. Hence $D_U \oplus D_U = 1/2$ and $D_A \oplus D_U = D_A/(1 + D_A)$. Note also that $D_A \oplus D_A = D_A/2$.

3.2 Subtraction ($\ominus$)

Let us define the N-adic Cantor fractal $C = A \ominus B$, by means of $\gamma_C = \gamma_A/\gamma_B$, resulting in

$$\frac{1}{D_C} = \frac{1}{D_A} - \frac{1}{D_B}, \qquad D_C = D_A \ominus D_B = \frac{D_A D_B}{D_B - D_A}.$$

This definition is consistent only for $\gamma_A < \gamma_B/N$, i.e., for

$$D_A < D_B \oplus D_U = \frac{D_B}{1 + D_B} < \frac{1}{2},$$

where $0 < D_B < 1$. Under these assumptions, the operator $\ominus$ is compatible with $\oplus$ since

$$(D_A \ominus D_B) \oplus D_B = D_A.$$

However, let us note that both $(D_A \oplus D_B) \ominus D_B$, and $D_A \ominus D_A$ are not properly defined, since $D_A \oplus D_B > D_B \oplus D_U$, and $D_A > D_A \oplus D_U$, respectively.

The operator $\ominus$ is neither commutative ($D_A \ominus D_B \neq D_B \ominus D_A$) nor associative

$$D_A \ominus (D_B \ominus D_C) = \frac{1}{\frac{1}{D_A} - \frac{1}{D_B} + \frac{1}{D_C}} \neq \frac{1}{\frac{1}{D_A} - \frac{1}{D_B} - \frac{1}{D_C}} = (D_A \ominus D_B) \ominus D_C.$$

The void set Z, with $\gamma_Z = 0$ and $D_Z = 0$, is also the absorbing element of $\ominus$, since $D_Z = D_A \ominus D_Z$ for all A.

3.3 Product ($\otimes$)

Let us define the N-adic Cantor fractal $C = A \otimes B$ by means of $\gamma_C = \gamma_A^{1/D_B}$, resulting in

$$\frac{1}{D_C} = -\frac{1}{D_B}\frac{\ln(\gamma_A)}{\ln(N)} = \frac{1}{D_A\, D_B}, \qquad D_C = D_A \otimes D_B = D_A\, D_B.$$

The correctness of this definition is straightforward since for $0 < \gamma_A, \gamma_B < 1/N$, $0 < \gamma_C < 1/N$; similarly, $0 < D_A, D_B < 1$ results in $0 < D_C < 1$.

The operator $\otimes$ is commutative ($D_A \otimes D_B = D_B \otimes D_A$) and associative

$$D_A \otimes (D_B \otimes D_C) = D_A\, D_B\, D_C = (D_A \otimes D_B) \otimes D_C.$$

It is also distributive with respect to the addition, since

$$D_A \otimes (D_B \oplus D_C) = \frac{D_A\, D_B\, D_C}{D_B + D_C} = (D_A \otimes D_B) \oplus (D_A \otimes D_C),$$

and also with respect to the subtraction, as

$$D_A \otimes (D_B \ominus D_C) = \frac{D_A\, D_B\, D_C}{D_C - D_B} = (D_A \otimes D_B) \ominus (D_A \otimes D_C).$$

Note that the operator $\oplus$ is not distributive with respect to the product, since

$$D_A \oplus (D_B \otimes D_C) = \frac{D_A\, D_B\, D_C}{D_A + D_B\, D_C} \neq \frac{D_A^2\, D_B\, D_C}{2\, D_A + D_B\, D_C} = (D_A \oplus D_B) \otimes (D_A \oplus D_C).$$

The operator $\ominus$ is also not distributive with respect to the product.

The void set Z is the absorbing element of $\otimes$, since $D_Z = D_A \otimes D_Z$ for all A, and the unit segment U is the unit element of the product, since $D_A \otimes D_U = D_A$.

3.4 Division ($\oslash$)

Let us define the N-adic Cantor fractal $C = A \oslash B$ by means of $\gamma_C = \gamma_A^{D_B} = \ln(\gamma_B)/\ln(\gamma_A)$, resulting in

$$\frac{1}{D_C} = -D_B \frac{\ln(\gamma_A)}{\ln(N)} = \frac{1}{D_B\, D_A}, \qquad D_C = D_A \oslash D_B = \frac{D_A}{D_B}.$$

The correctness of this definition requires that $0 < \gamma_A < \gamma_B < 1/N$, i.e., $0 < D_A < D_B < 1$. The division operator is compatible with the product, since

$$(D_A \oslash D_B) \otimes D_B = (D_A \otimes D_B) \otimes D_B = D_A,$$

and with the unit element $D_A \oslash D_U = D_A$.

The operator $\oslash$ is neither commutative ($D_A \oslash D_B \neq D_B \oslash D_A$) nor associative

$$D_A \oslash (D_B \oslash D_C) = \frac{D_A\, D_C}{D_B} \neq \frac{D_A}{D_B\, D_C} = (D_A \oslash D_B) \oslash D_C.$$

It is also distributive with respect to the product, since

$$D_A \oslash (D_B \otimes D_C) = \frac{D_A}{D_B\, D_C} = (D_A \oslash D_B) \otimes (D_A \oslash D_C),$$

but is not distributive with respect to either the addition nor the subtraction.

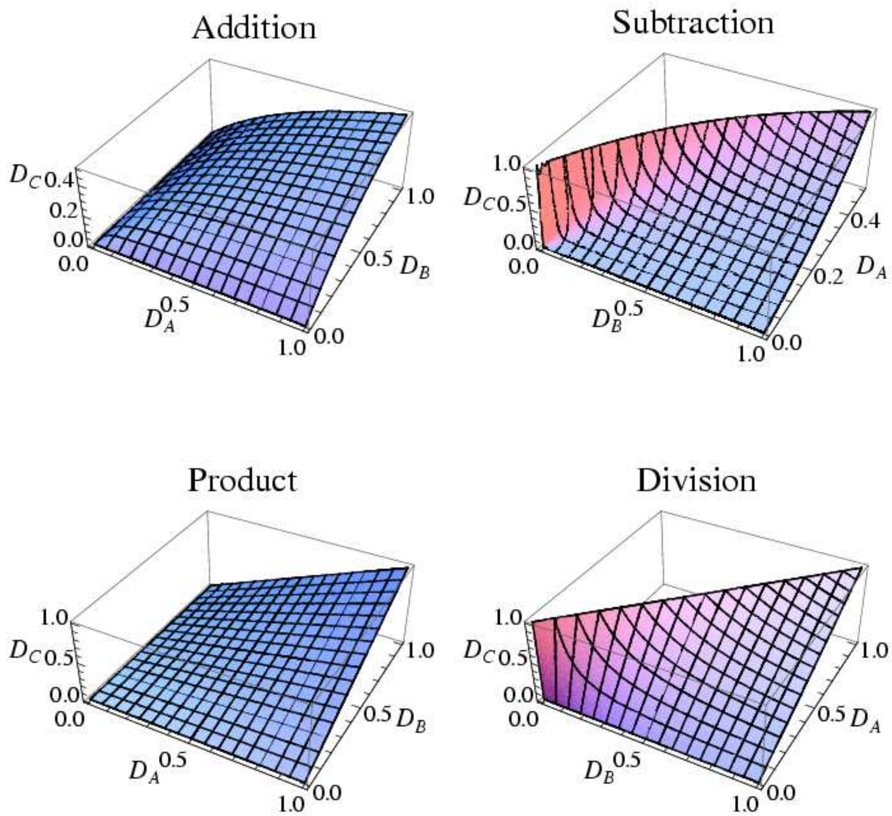

Fig. 2. Plots of the result of operations $D_A \oplus D_B$ (Addition), $D_A \ominus D_B$ (Subtraction), $D_A \otimes D_B$ (Product), and $D_A \oslash D_B$ (Division).

3.5 Integer power

Let us define the integer power of a N-adic Cantor fractal, $C = A^{(n)}$, where $n \geq 0$ is an integer, by means of $\gamma_C = \gamma_A^{1/D_A^{n-1}}$, yielding for the fractal dimension

$$D_A^{(n)} = D_A \otimes D_A \otimes \overset{(n)}{\cdots} \otimes D_A = (D_A)^n, \qquad \mathbb{N} \ni n \geq 0,$$

where, as usual, $D_A^{(0)} = D_U$, and $D_A^{(1)} = D_A$. The operator of integer power of fractal dimensions has the usual properties of the integer power of numbers inherated from those of the product $\otimes$.

3.6 Infinitesimal of fractal dimension

The development of an infinitesimal calculus for fractal dimension can be easily accomplished by using the arithmetical operators presented in previous sections and taking into account that an infinitesimal change of fractal dimension ruled by changes in the factor γ is given by

$$dD = -\frac{\ln(N)}{\gamma\ \ln^2(\gamma)}\, d\gamma.$$

This infinitesimal has only one parameter, being simpler than that introduced by Ref. [1] which requires the infinitesimal variation of two parameters.

4 Conclusions

Polyadic Cantor fractals may have any arbitrary dimension in the real interval $[0, 1]$ based on the continuity of their scaling factor γ. An algebraic structure that allows to develop an arithmetic with operators for addition, subtraction, product, and division of fractal dimension has been introduced. The new operators have the usual properties of the corresponding operations with real numbers. The introduction of an infinitesimal change of fractal dimension combined with these arithmetical operator allow the free manipulation of fractal dimension with the tools of calculus. Our results show that, even taking into account that fractal sets, by definition, are non-differentiable, it is possible to differentiate their dimension and manipulate it arithmetically.

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