Strong Approximation of Fractional Brownian Motion by Moving Averages of Simple Random Walks

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Dedicated to Pál Révész on the occasion of his 65th birthday

Abstract

The fractional Brownian motion is a generalization of ordinary Brownian motion, used particularly when long-range dependence is required. Its explicit introduction is due to B.B. Mandelbrot and J.W. van Ness (1968) as a self-similar Gaussian process $W^{(H)}(t)$ with stationary increments. Here self-similarity means that $(a^{-H}W^{(H)}(at) : t \geq 0) \stackrel{d}{=} (W^{(H)}(t) : t \geq 0)$, where $H \in (0, 1)$ is the Hurst parameter of fractional Brownian motion.

F.B. Knight gave a construction of ordinary Brownian motion as a limit of simple random walks in 1961. Later his method was simplified by P. Révész (1990) and then by the present author (1996). This approach is quite natural and elementary, and as such, can be extended to more general situations. Based on this, here we use moving averages of a suitable nested sequence of simple random walks that almost surely uniformly converge to fractional Brownian motion on compacts when $H \in (\frac{1}{4}, 1)$. The rate of convergence proved in this case is $O(N^{-\min(H-\frac{1}{4}, \frac{1}{4})} \log N)$, where N is the number of steps used for the approximation.

If the more accurate (but also more intricate) Komlós, Major, Tusnády (1975, 1976) approximation is used instead to embed random walks into ordinary Brownian motion, then the same type of moving averages almost surely uniformly converge to fractional Brownian motion on compacts for any $H \in (0, 1)$. Moreover, the convergence rate is conjectured to be the best possible $O(N^{-H} \log N)$, though only $O(N^{-\min(H, \frac{1}{2})} \log N)$ is proved here.

1

1 Fractional Brownian motion

The fractional Brownian motion (fBM) is a generalization of ordinary Brownian motion (BM) used particularly when long-range dependence is essential. Though the history of fBM can be traced back to [Kolmogorov (1940)] and others, its explicit introduction is due to [Mandelbrot and van Ness (1968)]. Their intention was to define a *self-similar*, centered Gaussian process $W^{(H)}(t)$ $(t \geq 0)$ with stationary but not independent increments and with continuous sample paths a.s. Here self-similarity means that for any $a > 0$,

$$\left(a^{-H} W^{(H)}(at) : t \geq 0 \right) \stackrel{d}{=} \left(W^{(H)}(t) : t \geq 0 \right), \tag{1}$$

where $H \in (0, 1)$ is the *Hurst parameter* of the fBM and $\stackrel{d}{=}$ denotes equality in distribution. They showed that these properties characterize fBM. The case $H = \frac{1}{2}$ reduces to ordinary BM with independent increments, while the cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$ give negatively, respectively, positively correlated increments, see [Mandelbrot and van Ness (1968)]. It seems that in the applications of fBM the case $H > \frac{1}{2}$ is the most frequently used.

[Mandelbrot and van Ness (1968)] gave the following explicit representation of fBM as a moving average of ordinary, but two-sided BM $W(s)$, $s \in \mathbb{R}$:

$$W^{(H)}(t) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_{-\infty}^{t} \left[(t - s)^{H - \frac{1}{2}} - (-s)_{+}^{H - \frac{1}{2}} \right] \, \mathrm{d}W(s) \qquad (t \geq 0), \tag{2}$$

where $(x)_+ = \max(x, 0)$.

The idea of (2) is related to *deterministic fractional calculus*, which has an even longer history than fBM, going back to Liouville, Riemann, and others, see in [Samko et al. (1993)]. Its simplest case is when a continuous function f and a positive integer α are given. Then an induction with integration by parts can show that

$$f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) \, \mathrm{d}s$$

is the order α iterated antiderivative (or order α integral) of f. On the other hand, this integral is well-defined for non-integer positive values of α as well, in which case it can be called a fractional integral of f.

So, heuristically, the main part of (2),

$$W_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} W'(s) \, \mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \, \mathrm{d}W(s)$$

is the order α integral of the (in ordinary sense non-existing) white noise process $W'(t)$. Thus the fBM $W^{(H)}(t)$ can be considered as a stationary-increment modification of the fractional integral $W_\alpha(t)$ of the white noise process, where $\alpha = H + \frac{1}{2} \in (\frac{1}{2}, \frac{3}{2})$.

2 Random walk construction of ordinary Brownian motion

It is interesting that a very natural and elementary construction of ordinary BM as a limit of random walks (RWs) appeared relatively late. The mathematical theory of BM began around 1900 with the works of Bachelier, Einstein,

Smoluchowski, and others. The first existence construction was given by Wiener (1921, 1923) that was followed by several others later. F.B. Knight (1961) introduced the first construction by random walks that was later simplified by P. Révész (1990). The present author was fortunate enough to hear this version of the construction directly from Pál Révész in a seminar at the Technical University of Budapest a couple of years before the publication of Révész's book in 1990 and got immediately fascinated by it. The result of an effort to further simplify it appeared in [Szabados (1996)]. From now on, the expression *RW construction* will always refer to the version discussed in the latter. It is asymptotically equivalent to applying [Skorohod (1965)] embedding to find a nested dyadic sequence of RWs in BM, see Theorem 4 in [Szabados (1996)]. As such, it has some advantages and disadvantages compared to the celebrated best possible approximation by BM of partial sums of random variables with moment generator function finite around the origin. The latter was obtained by Komlós, Major, and Tusnády (1975, 1976), and will be abbreviated *KMT approximation* in the sequel. The main advantages of the RW construction are that it is elementary, explicit, uses only past values to construct new ones, easy to implement in practice, and very suitable for approximating stochastic integrals, see Theorem 6 in [Szabados (1996)] and also [Szabados (1990)]. Recall that the KMT approximation constructs partial sums (e.g. a simple symmetric RW) from BM itself (or from an i.i.d. sequence of standard normal random variables) by an intricate sequence of conditional quantile transformations. To construct any new value it uses to whole sequence (past and future values as well). On the other hand, the major weakness of the RW construction is that it gives a rate of convergence $O(N^{-\frac{1}{4}} \log N)$, while the rate of the KMT approximation is the best possible $O(N^{-\frac{1}{2}} \log N)$, where N is the number of steps (terms) considered in the RW.

In the sequel first the main properties of the above-mentioned RW construction are summarized. Then this RW construction is used to define an approximation, similar to (2), of fBM by moving averages of the RW. The convergence and the error of this approximation are discussed next. As a consequence of the relatively weaker approximation properties of the RW construction, the convergence to fBM will be established only for $H \in (\frac{1}{4}, 1)$, and the rate of convergence will not be the best possible either. To compensate for this, at the end of the paper we discuss the convergence and error properties of a similar construction of fBM that uses the KMT approximation instead, which converges for all $H \in (0, 1)$ and whose convergence rate can be conjectured to be the best possible when approximating fBM by moving averages of RWs.

The RW construction of BM summarized here is taken from [Szabados (1996)]. We start with an infinite matrix of i.i.d. random variables $X_m(k)$,

$$\mathbf{P} \left\{ X_m(k) = 1 \right\} = \mathbf{P} \left\{ X_m(k) = -1 \right\} = \frac{1}{2} \qquad (m \geq 0, k \geq 1),$$

defined on the same underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Each row of this matrix is a basis of an approximation of BM with a certain dyadic step size $\Delta t = 2^{-2m}$ in time and a corresponding step size $\Delta x = 2^{-m}$ in space, illustrated by the next table.

The second step of the construction is *twisting*. From the independent random walks (i.e. from the rows of Table 1), we want to create dependent ones so that after shrinking temporal and spatial step sizes, each consecutive RW

Table 1: The starting setting for the RW construction of BM

Δt	Δx	i.i.d. sequence	RW
1	1	$X_0(1), X_0(2), X_0(3), \ldots$	$S_0(n) = \sum_{k=1}^{n} X_0(k)$
2^{-2}	2^{-1}	$X_1(1), X_1(2), X_1(3), \ldots$	$S_1(n) = \sum_{k=1}^{n} X_1(k)$
2^{-4}	2^{-2}	$X_2(1), X_2(2), X_2(3), \ldots$	$S_2(n) = \sum_{k=1}^{n} X_2(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

becomes a refinement of the previous one. Since the spatial unit will be halved at each consecutive row, we define stopping times by $T_m(0) = 0$, and for $k \geq 0$,

$$T_m(k+1) = \min \{n : n > T_m(k), |S_m(n) - S_m(T_m(k))| = 2\} \qquad (m \geq 1).$$

These are the random time instants when a RW visits even integers, different from the previous one. After shrinking the spatial unit by half, a suitable modification of this RW will visit the same integers in the same order as the previous RW. (This is what we call a refinement.) We will operate here on each point $\omega \in \Omega$ of the sample space separately, i.e. we fix a sample path of each RW appearing in Table 1. Thus each *bridge* $S_m(T_m(k+1)) - S_m(T_m(k))$ has to mimic the corresponding step $X_{m-1}(k+1)$ of the previous RW. We define twisted RWs $\tilde{S}_m$ recursively for $m = 1, 2, 3, \ldots$ using $\tilde{S}_{m-1}$, starting with $\tilde{S}_0(n) = S_0(n)$ ($n \geq 0$). With each fixed m we proceed for $k = 0, 1, 2, \ldots$ successively, and for every n in the corresponding bridge, $T_m(k) < n \leq T_m(k+1)$. Any bridge is flipped if its sign differs from the desired (Figs. 1-3):

$$\tilde{X}_m(n) = \begin{cases} X_m(n) & \text{if } S_m(T_m(k+1)) - S_m(T_m(k)) = 2\tilde{X}_{m-1}(k+1), \\ -X_m(n) & \text{otherwise,} \end{cases}$$

and then $\tilde{S}_m(n) = \tilde{S}_m(n-1) + \tilde{X}_m(n)$. Then each $\tilde{S}_m(n)$ ($n \geq 0$) is still a simple, symmetric RW, see Lemma 1 in [Szabados (1996)]. Moreover, the twisted RWs have the desired refinement property:

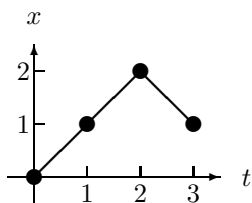$$\frac{1}{2}\tilde{S}_m(T_m(k)) = \tilde{S}_{m-1}(k) \qquad (m \geq 1, k \geq 0). \tag{3}$$

Figure 1: $B_0(t; \omega) = S_0(t; \omega)$.

The last step of the RW construction is *shrinking*. The sample paths of $\tilde{S}_m(n)$ ($n \geq 0$) can be extended to continuous functions by linear interpolation, this way one gets $\tilde{S}_m(t)$ ($t \geq 0$) for real t. Then we define the *mth approximation of BM* (see Fig. 4) by
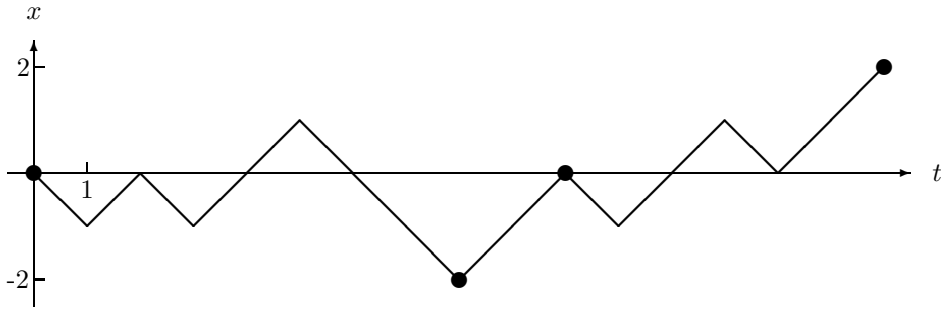
$$B_m(t) = 2^{-m}\tilde{S}_m(t2^{2m}). \tag{4}$$
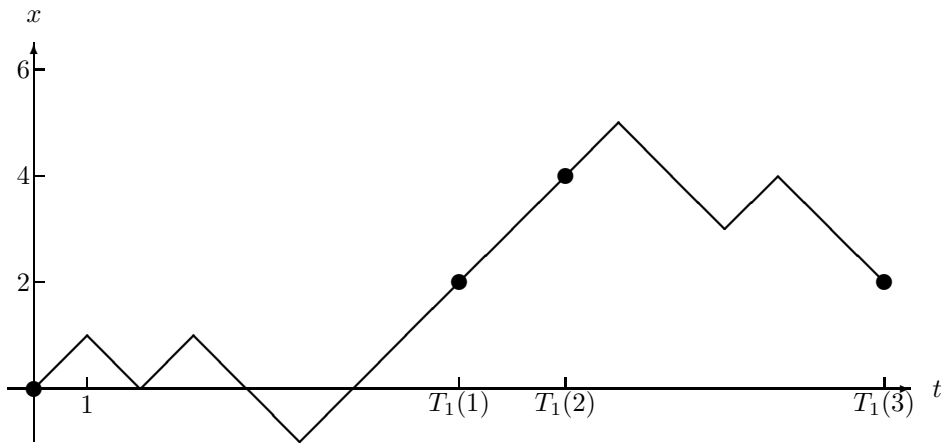
4

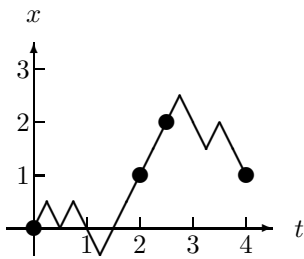Figure 2: $S_1(t; \omega)$.

Figure 3: $\tilde{S}_1(t; \omega)$.

Figure 4: $B_1(t; \omega)$.

Compare three steps of a sample path of the first approximation $B_0(t; \omega)$ and the corresponding part of the second approximation $B_1(t; \omega)$ on Figs. 1 and 4. The second visits the same integers (different from the previous one) in the same order as the first, so mimics the first, but the corresponding time instants differ in general: $2^{-2}T_1(k) \neq k$. Similarly, (3) implies the general *refinement property*

$$B_{m+1}\left(T_{m+1}(k)\, 2^{-2(m+1)}\right) = B_m\left(k2^{-2m}\right) \qquad (m \geq 0, k \geq 0), \qquad (5)$$

5

but there is a *time lag*

$$T_{m+1}(k)\ 2^{-2(m+1)} - k2^{-2m} \neq 0 \tag{6}$$

in general.

The basic idea of the RW construction of BM is that these time lags become uniformly small if m gets large enough. It can be proved by the following simple lemma.

Lemma 1. *Suppose that $X_1, X_2, \ldots, X_N$ is an i.i.d. sequence of random variables, $\mathbf{E}(X_k) = 0$, $\mathbf{Var}(X_k) = 1$, and their moment generating function $\mathbf{E}\left(e^{uX_k}\right)$ is finite for $|u| \leq u_0$, $u_0 > 0$. Let $S_j = X_1 + \cdots + X_j$, $1 \leq j \leq N$. Then for any $C > 1$ and $N \geq N_0(C)$ one has*

$$\mathbf{P}\left\{\max_{0 \leq j \leq N} |S_j| \geq (2CN \log N)^{\frac{1}{2}}\right\} \leq 2N^{1-C}.$$

This basic fact follows from a large deviation inequality, see e.g. Section XVI,6 in [Feller (1966)]. Lemma 1 easily implies the uniform smallness of time lags in (6).

Lemma 2. *For any $K > 0$, $C > 1$, and for any $m \geq m_0(C)$, we have*

$$\mathbf{P}\left\{\max_{0 \leq k2^{-2m} \leq K} \left|T_{m+1}(k)\ 2^{-2(m+1)} - k2^{-2m}\right| \geq \left(\frac{3}{2}CK \log_* K\right)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m}\right\}$$
$$\leq 2\left(K2^{2m}\right)^{1-C},$$

where $\log_*(x) = \max(1, \log x)$.

Not surprisingly, this and the refinement property (5) imply the uniform closeness of two consecutive approximations of BM if m is large enough.

Lemma 3. *For any $K > 0$, $C > 1$, and $m \geq m_1(C)$, we have*

$$\mathbf{P}\left\{\max_{0 \leq k2^{-2m} \leq K} \left|B_{m+1}(k2^{-2m}) - B_m(k2^{-2m})\right| \geq K^{\frac{1}{4}}(\log_* K)^{\frac{3}{4}} m2^{-\frac{m}{2}}\right\}$$
$$\leq 3(K2^{2m})^{1-C}.$$

This lemma ensures the a.s. uniform convergence of the RW approximations on compact intervals and it is clear that the limit process is the Wiener process (BM) with continuous sample paths almost surely.

Theorem 1. *The RW approximation $B_m(t)$ $(t \geq 0, m = 0, 1, 2, \ldots)$ a.s. uniformly converges to a Wiener process $W(t)$ $(t \geq 0)$ on any compact interval $[0, K], K > 0$. For any $K > 0$, $C \geq 3/2$, and for any $m \geq m_2(C)$, we have*

$$\mathbf{P}\left\{\max_{0 \leq t \leq K} |W(t) - B_m(t)| \geq K^{\frac{1}{4}}(\log_* K)^{\frac{3}{4}} m2^{-\frac{m}{2}}\right\} \leq 6(K2^{2m})^{1-C}.$$

The results quoted above correspond to Lemmas 2-4 and Theorem 3 in [Szabados (1996)]. We mention that the statements presented here are given in somewhat sharper forms, but they can be read easily from the proofs in the above reference.

3 A pathwise approximation of fractional Brownian motion

An almost surely convergent pathwise construction of fBM was given by Carmona and Coutin (1998) representing fBM as a linear functional of an infinite dimensional Gaussian process. Another pathwise construction was given by Decreusefond and Üstünel (1998, 1999) which converges in the L^2 sense. This construction uses discrete approximations of the moving average representation of fBM (2), based on deterministic partitions of the time axis. More exactly, (2) is substituted by an integral over the compact interval $[0, t]$, but with a more complicated kernel containing a hypergeometric function too.

The approximation of fBM discussed here will also be a discrete version of the moving average representation (2) of fBM, but dyadic partitions are taken on the spatial axis of BM and so one gets random partitions on the time axis. This is asymptotically a Skorohod-type embedding of nested RWs into BM. As a result, instead of integral we have sum, and BM is substituted by the nested, refining sequence of its RW approximations discussed in the previous section. Since (2) contains two-sided BM, we need two such sequences: one for the right and one for the left half-axis. From now on, we are going to use the following notations: $m \geq 0$ is an integer, $\Delta t = 2^{-2m}$, $t_x = x\Delta t$ ($x \in \mathbb{R}$). Then by definition, the *mth approximation of fBM* is: $B_m^{(H)}(0) = 0$, and for positive integers k,

$$B_m^{(H)}(t_k)$$

$$= \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k-1} \left[(t_k - t_r)^{H-\frac{1}{2}} - (-t_r)_+^{H-\frac{1}{2}} \right] \left[B_m(t_r + \Delta t) - B_m(t_r) \right]$$

$$= \frac{2^{-2Hm}}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k-1} \left[(k - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] \tilde{X}_m(r+1), \quad (7)$$

where the convention $0^{H-\frac{1}{2}} = 0$ is applied even for negative exponents.

$B_m^{(H)}$ is *well-defined*, since the "infinite part"

$$\sum_{r=-\infty}^{-1} \left[(k - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \right] \tilde{X}_m(r+1) =: \sum_{r=-\infty}^{-1} Y_{k,-r}$$

converges a.s. to a random variable Z_k by Kolmogorov's "three-series theorem": $\mathbf{E}(Y_{k,v}) = 0$ and

$$\sum_{v=1}^{\infty} \mathbf{Var}(Y_{k,v}) = \sum_{v=1}^{\infty} v^{2H-1} \left[\left(1 + \frac{k}{v} \right)^{H-\frac{1}{2}} - 1 \right]^2 \sim \sum_{v=1}^{\infty} \frac{\mathrm{const}}{v^{3-2H}} < \infty.$$

It is useful to write $B_m^{(H)}$ in another form applying a discrete version of integration by parts. Starting with (7) and rearranging it according to $B_m(t_r)$, one obtains for $k \geq 1$ that

$$B_m^{(H)}(t_k) = \sum_{r=-\infty}^{k} \frac{h(t_r - \Delta t, t_k) - h(t_r, t_k)}{\Delta t} B_m(t_r) \Delta t, \tag{8}$$

where we introduced the kernel

$$h(s,t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left[(t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] \qquad (s \le t). \tag{9}$$

This way we have got a discrete version of

$$W^{(H)}(t) = \frac{-1}{\Gamma\left(H+\frac{1}{2}\right)} \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\left[(t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] W(s)\,\mathrm{d}s, \tag{10}$$

which is what one obtains from (2) using a formal integration by parts (cf. Lemma 5 below).

To support the above definition we show that $B_m^{(H)}$ has properties analogous to the characterizing properties of fBM in a discrete setting.

(a) $B_m^{(H)}$ is *centered* (clear from its definition) and has *stationary increments*: if k_0 and k are non-negative integers, then (substituting $u = r - k_0$)

$$
\begin{aligned}
& B_m^{(H)}(t_{k_0} + t_k) - B_m^{(H)}(t_{k_0}) \\
&= \frac{2^{-2Hm}}{\Gamma\left(H+\frac{1}{2}\right)} \left\{ \sum_{r=0}^{k_0+k-1} (k_0 + k - r)^{H-\frac{1}{2}} \tilde{X}_m(r+1) \right. \\
&\qquad - \sum_{r=0}^{k_0} (k_0 - r)^{H-\frac{1}{2}} \tilde{X}_m(r+1) \\
&\qquad \left. + \sum_{r=-\infty}^{-1} \left[(k_0 + k - r)^{H-\frac{1}{2}} - (k_0 - r)^{H-\frac{1}{2}} \right] \tilde{X}_m(r+1) \right\} \\
&= \frac{2^{-2Hm}}{\Gamma\left(H+\frac{1}{2}\right)} \left\{ \sum_{u=-k_0}^{k-1} (k - u)^{H-\frac{1}{2}} \tilde{X}_m(k_0 + u + 1) \right. \\
&\qquad - \sum_{u=-k_0}^{0} (-u)^{H-\frac{1}{2}} \tilde{X}_m(k_0 + u + 1) \\
&\qquad \left. + \sum_{u=-\infty}^{-k_0-1} \left[(k - u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] \tilde{X}_m(k_0 + u + 1) \right\} \\
&= \frac{2^{-2Hm}}{\Gamma\left(H+\frac{1}{2}\right)} \sum_{u=-\infty}^{k-1} \left[(k - u)^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right] \tilde{X}_m(k_0 + u + 1) \overset{d}{=} B_m^{(H)}(t_k).
\end{aligned}
$$

(b) $B_m^{(H)}$ is approximately *self-similar* in the following sense. If $a = 2^{2m_0}$, where m_0 is an integer, $m_0 \ge -m$, then for any k non-negative integer for which ka is also an integer one has that

$$
\begin{aligned}
& a^{-H} B_m^{(H)}(ak2^{-2m}) \\
&= \frac{a^{-H} 2^{-2Hm}}{\Gamma\left(H+\frac{1}{2}\right)} \sum_{r=-\infty}^{ak-1} \left[(ak - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] \tilde{X}_m(r+1) \\
&= \frac{2^{-2H(m+m_0)}}{\Gamma\left(H+\frac{1}{2}\right)} \sum_{r=-\infty}^{k2^{2m_0}-1} \left[(k2^{2m_0} - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] \tilde{X}_m(r+1) \\
&\overset{d}{=} B_{m+m_0}^{(H)}(k2^{-2m}).
\end{aligned}
$$

8

On the other hand, Lemma 4 (and Theorem 2) below show that $B_m^{(H)}$ and $B_{m+1}^{(H)}$ (and $B_{m+n}^{(H)}$) are uniformly close with arbitrary large probability on any compact interval if m is large enough (when $H > \frac{1}{4}$). It could be proved in a similar fashion that for $a = j$, where $j \geq 0$ is an arbitrary integer, $2^{2n} \leq j \leq 2^{2(n+1)}$ with an integer $n \geq 0$, the finite dimensional distributions of

$$a^{-H} B_m^{(H)}(ak2^{-2m}) = \frac{2^{-H(2m+\log_2 j)}}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{jk-1} \left[(jk-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right] \tilde{X}_m(r+1)$$

can be made arbitrarily close to the the finite dimensional distributions of $B_{m+n}^{(H)}$ if m is large enough. Consequently, $B_m^{(H)}$ is arbitrarily close to self-similar for any dyadic $a = j2^{2m_0}$ if m is large enough.

(c) For any $0 < t_1 < \cdots < t_n$, the limit distribution of the vector

$$\left(B_m^{(H)}(t_1^{(m)}), B_m^{(H)}(t_2^{(m)}), \ldots, B_m^{(H)}(t_n^{(m)})\right)$$

as $m \to \infty$ is *Gaussian*, where $t_j^{(m)} = \lfloor t_j 2^{2m}\rfloor 2^{-2m}$, $1 \leq j \leq n$. This fact follows from Theorem 2 (based on Lemma 5) below that states that the process $B_m^{(H)}$ almost surely converges to the Gaussian process $W^{(H)}$ on compact intervals.

4   Convergence of the approximation to fBM

At first it will be shown that two consecutive approximations of fBM defined by (7), or equivalently by (8), are uniformly close if m is large enough, supposing $H > \frac{1}{4}$. Apparently, the above RW approximation of BM is not good enough to have convergence for $H \leq \frac{1}{4}$.

When proving convergence, a large deviation inequality similar to Lemma 1 will play an important role. If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables, $\mathbf{P}\{X_k = \pm 1\} = \frac{1}{2}$, and $S = \sum_r a_r X_r$, where not all $a_r \in \mathbb{R}$ are zero and $\mathbf{Var}(S) = \sum_r a_r^2 < \infty$, then

$$\mathbf{P}\left\{|S| \geq x\left(\mathbf{Var}(S)\right)^{\frac{1}{2}}\right\} \leq 2e^{-\frac{x^2}{2}} \qquad (x \geq 0), \tag{11}$$

see e.g. p. 33 in [Stroock (1993)]. The summation above may extend either to finitely many or to countably many terms.

As a corollary, if $S_1, S_2, \ldots, S_N$ are arbitrary sums of the above type, one can get the following analog of Lemma 1. For any $C > 1$ and $N \geq 1$,

$$\mathbf{P}\left\{\max_{1\leq k\leq N} |S_k| \geq (2C \log N)^{\frac{1}{2}} \max_{1\leq k\leq N} \left(\mathbf{Var}(S_k)\right)^{\frac{1}{2}}\right\}$$

$$\leq \sum_{k=1}^{N} \mathbf{P}\left\{|S_k| \geq (2C \log N \, \mathbf{Var}(S_k))^{\frac{1}{2}}\right\} \leq 2Ne^{-C \log N} = 2N^{1-C}. \tag{12}$$

Lemma 4. *For any* $H \in (\frac{1}{4}, 1)$, $K > 0$, $C \geq 3$, *and* $m \geq m_3(C)$, *we have*

$$\mathbf{P}\left\{\max_{0\leq t_k\leq K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| \geq \alpha(H, K)m2^{-\beta(H)m}\right\} \leq 8(K2^{2m})^{1-C},$$

9

where $t_k = k2^{-2m}$ for $k \geq 0$ integers, $\beta(H) = \min(2H - \frac{1}{2}, \frac{1}{2})$ and

$$\alpha(H, K) = \frac{(\log_* K)^{\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)} \left[\frac{\left|H - \frac{1}{2}\right|}{(1 - H)^{\frac{1}{2}}} + (\log_* K)^{\frac{1}{4}} \left(8K^{\frac{1}{4}} + 36 \left|H - \frac{1}{2}\right| K^{H - \frac{1}{4}} \right) \right]$$

if $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$,

$$\alpha(H, K) = \frac{(\log_* K)^{\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)} \left[\frac{\left|H - \frac{1}{2}\right|}{(1 - H)^{\frac{1}{2}}} + (\log_* K)^{\frac{1}{4}} \left(5 + 312 \left|H - \frac{1}{2}\right| \right) K^{H - \frac{1}{4}} \right]$$

if $H \in \left(\frac{1}{2}, 1\right)$. (The case $H = \frac{1}{2}$ is described by Lemma 3.)

Proof. The proof is long, but elementary. Introduce the following abbreviations: $\Delta B_m(t) = B_m(t + \Delta t) - B_m(t)$, $\Delta B_{m+1}(t) = B_{m+1}(t + \frac{1}{4}\Delta t) - B_{m+1}(t)$. Using (7) and then substituting $u = 4r + j$, one gets that

$$
\begin{aligned}
B_{m+1}^{(H)}(t_k) &= B_{m+1}^{(H)}(4k2^{-2(m+1)}) \\
&= \frac{2^{-2H(m+1)}}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{u=-\infty}^{4k-1} \left[(4k - u)^{H - \frac{1}{2}} - (-u)_+^{H - \frac{1}{2}} \right] \tilde{X}_{m+1}(u + 1) \\
&= \frac{2^{-2Hm-1}}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left[\left(k - r - \frac{j}{4}\right)^{H - \frac{1}{2}} - \left(-r - \frac{j}{4}\right)_+^{H - \frac{1}{2}} \right] \\
&\quad \times \tilde{X}_{m+1}(4r + j + 1) \\
&= \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left[(t_k - t_{r+\frac{j}{4}})^{H - \frac{1}{2}} - (-t_{r+\frac{j}{4}})_+^{H - \frac{1}{2}} \right] \Delta B_{m+1}(t_{r+\frac{j}{4}}).
\end{aligned}
$$

So, subtracting and adding a suitable "intermediate" term, one arrives at

$$
\begin{aligned}
\Gamma\left(H + \frac{1}{2}\right) &\left[B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k) \right] \\
&= \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left[(t_k - t_{r+\frac{j}{4}})^{H - \frac{1}{2}} - (-t_{r+\frac{j}{4}})_+^{H - \frac{1}{2}} \right] \Delta B_{m+1}(t_{r+\frac{j}{4}}) \\
&\quad - \left[(t_k - t_r)^{H - \frac{1}{2}} - (-t_r)_+^{H - \frac{1}{2}} \right] \frac{1}{4} \Delta B_m(t_r) \\
&= \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left\{ \left[(t_k - t_{r+\frac{j}{4}})^{H - \frac{1}{2}} - (-t_{r+\frac{j}{4}})_+^{H - \frac{1}{2}} \right] \right. \\
&\quad \left. - \left[(t_k - t_r)^{H - \frac{1}{2}} - (-t_r)_+^{H - \frac{1}{2}} \right] \right\} \Delta B_{m+1}(t_{r+\frac{j}{4}}) \\
&\quad + \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left[(t_k - t_r)^{H - \frac{1}{2}} - (-t_r)_+^{H - \frac{1}{2}} \right] \left[\Delta B_{m+1}(t_{r+\frac{j}{4}}) - \frac{1}{4} \Delta B_m(t_r) \right] \\
&=: \quad (Z_{m,k} + Y_{m,k} + V_{m,k} + U_{m,k}). \tag{13}
\end{aligned}
$$

Here we introduced the following notations:

$$Z_{m,k} = \sum_{r=0}^{k-1} \sum_{j=0}^{3} \left[(t_k - t_{r+\frac{j}{4}})^{H-\frac{1}{2}} - (t_k - t_r)^{H-\frac{1}{2}} \right] \Delta B_{m+1}(t_{r+\frac{j}{4}})$$

$$= 2^{-2Hm-1} \sum_{r=0}^{k-1} \sum_{j=0}^{3} \left[(k - r - \frac{j}{4})^{H-\frac{1}{2}} - (k - r)^{H-\frac{1}{2}} \right] \tilde{X}_{m+1}(4r + j + 1) \quad (14)$$

and

$$
\begin{aligned}
Y_{m,k} &= \sum_{r=0}^{k-1} (t_k - t_r)^{H-\frac{1}{2}} \sum_{j=0}^{3} \left[\Delta B_{m+1}(t_{r+\frac{j}{4}}) - \frac{1}{4} \Delta B_m(t_r) \right] \\
&= \sum_{r=0}^{k-1} (t_k - t_r)^{H-\frac{1}{2}} \left\{ [B_{m+1}(t_{r+1}) - B_{m+1}(t_r)] - [B_m(t_{r+1}) - B_m(t_r)] \right\} \\
&= \sum_{r=0}^{k} \left[(t_k - t_{r-1})^{H-\frac{1}{2}} - (t_k - t_r)^{H-\frac{1}{2}} \right] [B_{m+1}(t_r) - B_m(t_r)], \quad (15)
\end{aligned}
$$

applying "summation by parts" in the last row, as in (8). Similarly, we introduced the following notations for the corresponding "infinite parts" in (13) (using $v = -r$):

$$V_{m,k}$$

$$= 2^{-2Hm-1} \sum_{v=1}^{\infty} \sum_{j=0}^{3} \left[(k + v - \frac{j}{4})^{H-\frac{1}{2}} - (v - \frac{j}{4})^{H-\frac{1}{2}} - (k + v)^{H-\frac{1}{2}} + v^{H-\frac{1}{2}} \right]$$

$$\times \tilde{X}_{m+1}(-4v + j + 1), \quad (16)$$

and

$$
\begin{aligned}
U_{m,k} &= \sum_{v=1}^{\infty} \left[(t_k + t_v)^{H-\frac{1}{2}} - t_v^{H-\frac{1}{2}} \right] \sum_{j=0}^{3} \left[\Delta B_{m+1}(-t_{v-\frac{j}{4}}) - \frac{1}{4} \Delta B_m(-t_v) \right] \\
&= \sum_{v=1}^{\infty} \left[(t_k + t_v)^{H-\frac{1}{2}} - t_v^{H-\frac{1}{2}} \right] \\
&\quad \times \left\{ [B_{m+1}(-t_{v-1}) - B_{m+1}(-t_v)] - [B_m(-t_{v-1}) - B_m(-t_v)] \right\} \\
&= \sum_{v=1}^{\infty} \left[(t_k + t_{v+1})^{H-\frac{1}{2}} - t_{v+1}^{H-\frac{1}{2}} - (t_k + t_v)^{H-\frac{1}{2}} + t_v^{H-\frac{1}{2}} \right] \\
&\quad \times [B_{m+1}(-t_v) - B_m(-t_v)]. \quad (17)
\end{aligned}
$$

The maxima of $Z_{m,k}, Y_{m,k}, V_{m,k}$ and $U_{m,k}$ can be estimated separately:

$$\max_{0 \le t_k \le K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| \quad (18)$$

$$\le \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \left(\max_k |Z_{m,k}| + \max_k |Y_{m,k}| + \max_k |V_{m,k}| + \max_k |U_{m,k}| \right),$$

where each maximum on the right hand side is taken for $1 \le k \le K2^{2m}$ and one can suppose that $K2^{2m} \ge 1$, that is, $\Delta t \le K$, since otherwise the maximal difference in (19) is zero.

11

(a) The maximum of $Z_{m,k}$. In the present case the large deviation inequality (11), or rather, its corollary (12) is applied. By (14),

$$
\begin{aligned}
\mathbf{Var}(Z_{m,k}) &= 2^{-4Hm-2} \sum_{r=0}^{k-1} \sum_{j=0}^{3} \left[(k-r-\tfrac{j}{4})^{H-\frac{1}{2}} - (k-r)^{H-\frac{1}{2}} \right]^2 \\
&= 2^{-4Hm-2} \sum_{r=0}^{k-1} \sum_{j=0}^{3} (k-r)^{2H-1} \left[(1 - \tfrac{j}{4(k-r)})^{H-\frac{1}{2}} - 1 \right]^2.
\end{aligned}
$$

The term in brackets can be estimated using a binomial series with $0 \le j \le 3$, $k-r \ge 1$:

$$
\begin{aligned}
\left| \left(1 - \frac{j}{4(k-r)} \right)^{H-\frac{1}{2}} - 1 \right| &= \left| \sum_{s=1}^{\infty} \binom{H-\frac{1}{2}}{s} (-1)^s \left(\frac{j}{4(k-r)} \right)^s \right| \\
&\le \left| H - \frac{1}{2} \right| \frac{j}{4(k-r)} \left(1 - \frac{j}{4(k-r)} \right)^{-1} \le \left| H - \frac{1}{2} \right| \frac{j}{4(k-r)} \left(1 - \frac{j}{4} \right)^{-1}.
\end{aligned}
$$

Thus

$$
\sum_{j=0}^{3} \left[\left(1 - \frac{j}{4(k-r)} \right)^{H-\frac{1}{2}} - 1 \right]^2 \le \left(H - \frac{1}{2} \right)^2 \frac{91}{9} \frac{1}{(k-r)^2}.
$$

Also, if $k \ge 1$, one has

$$
\sum_{r=0}^{k-1} (k-r)^{2H-3} < 1 + \int_0^{k-1} (k-x)^{2H-3} \, dx \le \frac{3}{2} \frac{1}{1-H}.
$$

Then for any $k \ge 1$ it follows that

$$
\mathbf{Var}(Z_{m,k}) \le 2^{-4Hm} \left(H - \frac{1}{2} \right)^2 \frac{273}{72} \frac{1}{1-H}.
$$

Hence taking $N = K2^{2m}$ and $C > 1$ in (12), one obtains that

$$
\begin{aligned}
\mathbf{P} &\left\{ \max_{1 \le k \le N} |Z_{m,k}| \ge \left(\frac{273}{72} \right)^{\frac{1}{2}} \left| H - \frac{1}{2} \right| (1-H)^{-\frac{1}{2}} 2^{-2Hm} (2C \log N)^{\frac{1}{2}} \right\} \\
&\le \sum_{k=1}^{N} \mathbf{P} \left\{ |Z_{m,k}| \ge (2C \log N \, \mathbf{Var}(Z_{m,k}))^{\frac{1}{2}} \right\} \le 2N^{1-C}.
\end{aligned}
$$

Since
$$
\log N = \log \left(K2^{2m} \right) \le (1 + \log 4) m \log_* K \le 2.5 m \log_* K, \tag{19}
$$

one obtains the following result:

$$
\max_{1 \le k \le K2^{2m}} |Z_{m,k}| \le 5 \left| H - \frac{1}{2} \right| (1-H)^{-\frac{1}{2}} (C \log_* K)^{\frac{1}{2}} \, m^{\frac{1}{2}} 2^{-2Hm}, \tag{20}
$$

with the exception of a set of probability at most $2 \left(K2^{2m} \right)^{1-C}$, where $m \ge 1$, $K > 0$ and $C > 1$ are arbitrary.

(b) The maximum of $Y_{m,k}$. By its definition (15),

$$\max_{1\leq k\leq K2^{2m}}|Y_{m,k}| \leq \max_{0\leq t_r\leq K}|B_{m+1}(t_r)-B_m(t_r)|$$
$$\times\max_{0\leq t_k\leq K}\sum_{0\leq t_r\leq t_k}\left|(t_k-t_{r-1})^{H-\frac{1}{2}}-(t_k-t_r)^{H-\frac{1}{2}}\right|.$$

The first factor, the maximal difference between two consecutive approximations of BM appearing here can be estimated by Lemma 3. For the second factor one can apply a binomial series:

$$\sum_{r=0}^{k}\left|(t_k-t_{r-1})^{H-\frac{1}{2}}-(t_k-t_r)^{H-\frac{1}{2}}\right|$$
$$= 2^{-m(2H-1)}\left\{1+\left|2^{H-\frac{1}{2}}-1\right|+\sum_{r=0}^{k-2}(k-r)^{H-\frac{1}{2}}\left|\left(1+\frac{1}{k-r}\right)^{H-\frac{1}{2}}-1\right|\right\}$$
$$= 2^{-m(2H-1)}\left\{1+\left|2^{H-\frac{1}{2}}-1\right|+\sum_{r=0}^{k-2}(k-r)^{H-\frac{1}{2}}\left|\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{1}{(k-r)^s}\right|\right\}$$
$$\leq 2^{-m(2H-1)}\left\{1+\left|2^{H-\frac{1}{2}}-1\right|+\sum_{r=0}^{k-2}(k-r)^{H-\frac{1}{2}}\frac{|H-\frac{1}{2}|}{k-r}\right\}.$$

Since for $H\neq\frac{1}{2}$

$$\sum_{r=0}^{k-2}(k-r)^{H-\frac{3}{2}}\leq\int_0^{k-1}(k-x)^{H-\frac{3}{2}}\,\mathrm{d}x=\frac{1-k^{H-\frac{1}{2}}}{\frac{1}{2}-H},$$

it follows for any $m\geq 0$ that

$$\max_{1\leq k\leq K2^{2m}}\sum_{r=0}^{k}\left|(t_k-t_{r-1})^{H-\frac{1}{2}}-(t_k-t_r)^{H-\frac{1}{2}}\right|$$
$$\leq 2^{-m(2H-1)}\max_{1\leq k\leq K2^{2m}}\left\{1+\left|2^{H-\frac{1}{2}}-1\right|+\left|1-k^{H-\frac{1}{2}}\right|\right\}$$
$$\leq \begin{cases} 2^{-2m(H-\frac{1}{2})}\left(3-2^{H-\frac{1}{2}}\right)\leq 3\,2^{-2m(H-\frac{1}{2})} & \text{if } 0<H<\frac{1}{2}, \\ 2^{-2m(H-\frac{1}{2})}\left(2^{H-\frac{1}{2}}-1\right)+K^{H-\frac{1}{2}}\leq(2K)^{H-\frac{1}{2}} & \text{if } \frac{1}{2}<H<1. \end{cases}$$

(In the last row we used that here $2^{-2m}\leq K$.)

Combining this with Lemma 3, we obtain the result

$$\max_{1\leq k\leq K2^{2m}}|Y_{m,k}|\leq\begin{cases} 3K^{\frac{1}{4}}(\log_* K)^{\frac{3}{4}}m2^{-2m\left(H-\frac{1}{4}\right)} & \text{if } \frac{1}{4}<H<\frac{1}{2}, \\ 2^{H-\frac{1}{2}}K^{H-\frac{1}{4}}(\log_* K)^{\frac{3}{4}}m2^{-\frac{m}{2}} & \text{if } \frac{1}{2}<H<1. \end{cases} \tag{21}$$

with the exception of a set of probability at most $3\left(K2^{2m}\right)^{1-C}$, where $K>0$, $C>1$ are arbitrary, and $m\geq m_1(C)$. Thus in the case $0<H<\frac{1}{2}$ we have only a partial result: the relative weakness of the above-described RW approximation of BM causes that apparently we have no convergence for $0<H\leq\frac{1}{4}$.

(c) The maximum of $V_{m,k}$. Here one can use the same idea as in part (a), including the application of the corollary (12) of the large deviation principle. We begin with (16),

$\mathbf{Var}(V_{m,k})$

$$= 2^{-4Hm-2} \sum_{v=1}^{\infty} \sum_{j=0}^{3} \left[(k+v-\tfrac{j}{4})^{H-\frac{1}{2}} - (k+v)^{H-\frac{1}{2}} - (v-\tfrac{j}{4})^{H-\frac{1}{2}} + v^{H-\frac{1}{2}} \right]^2$$

$$= 2^{-4Hm-2} \sum_{v=1}^{\infty} \sum_{j=0}^{3} \left\{ (k+v)^{H-\frac{1}{2}} \left[\left(1 - \frac{j}{4(k+v)} \right)^{H-\frac{1}{2}} - 1 \right] \right.$$

$$\left. - v^{H-\frac{1}{2}} \left[\left(1 - \frac{j}{4v} \right)^{H-\frac{1}{2}} - 1 \right] \right\}^2.$$

As in (a), now we use binomial series for the expressions in brackets ($k \geq 1$, $0 \leq j \leq 3$, $v \geq 1$):

$$A = (k+v)^{H-\frac{1}{2}} \left[\left(1 - \frac{j}{4(k+v)} \right)^{H-\frac{1}{2}} - 1 \right]$$

$$= (k+v)^{H-\frac{1}{2}} \sum_{s=1}^{\infty} \binom{H-\frac{1}{2}}{s} (-1)^s \left(\frac{j}{4(k+v)} \right)^s$$

and

$$B = v^{H-\frac{1}{2}} \left[\left(1 - \frac{j}{4v} \right)^{H-\frac{1}{2}} - 1 \right] = v^{H-\frac{1}{2}} \sum_{s=1}^{\infty} \binom{H-\frac{1}{2}}{s} (-1)^s \left(\frac{j}{4v} \right)^s.$$

Then A and B have the same sign and $0 \leq A_1 \leq |A| \leq |B| \leq B_2$, where

$$A_1 = (k+v)^{H-\frac{1}{2}} \left| H - \frac{1}{2} \right| \frac{j}{4(k+v)} = \left| H - \frac{1}{2} \right| \frac{j}{4} (k+v)^{H-\frac{3}{2}}$$

and

$$B_2 = v^{H-\frac{1}{2}} \left| H - \frac{1}{2} \right| \frac{j}{4v} \left(1 - \frac{j}{4v} \right)^{-1} \leq \left| H - \frac{1}{2} \right| \frac{j}{4-j} v^{H-\frac{3}{2}}.$$

Hence

$$(A-B)^2 \leq A_1^2 + B_2^2 \leq \left(H - \frac{1}{2} \right)^2 \left[\left(\frac{j}{4} \right)^2 (k+v)^{2H-3} + \left(\frac{j}{4-j} \right)^2 v^{2H-3} \right].$$

Since for any $k \geq 0$,

$$\sum_{v=1}^{\infty} (k+v)^{2H-3} < 1 + \int_1^{\infty} (k+x)^{2H-3} \, \mathrm{d}x = 1 + \frac{(k+1)^{2H-2}}{2-2H} < \frac{3}{2} \frac{1}{1-H}$$

it follows that

$\mathbf{Var}(V_{m,k})$

$$\leq 2^{-4Hm-2} \left(H - \frac{1}{2} \right)^2 \sum_{v=1}^{\infty} \sum_{j=0}^{3} \left[\left(\frac{j}{4} \right)^2 (k+v)^{2H-3} + \left(\frac{j}{4-j} \right)^2 v^{2H-3} \right]$$

$$\leq \frac{791}{192} \frac{\left(H - \frac{1}{2} \right)^2}{1-H} 2^{-4Hm}.$$

14

Applying corollary (12) of the large deviation inequality with $N = K2^{2m}$ one obtains that

$$\mathbf{P}\left\{\max_{1\le k\le N}|V_{m,k}| \ge \left(\frac{791}{192}\right)^{\frac{1}{2}}\left|H - \frac{1}{2}\right|(1-H)^{-\frac{1}{2}}2^{-2Hm}(2C\log N)^{\frac{1}{2}}\right\}$$

$$\le \sum_{k=1}^{N}\mathbf{P}\left\{|V_{m,k}| \ge (2C\log N\,\mathbf{Var}(V_{m,k}))^{\frac{1}{2}}\right\} \le 2N^{1-C}.$$

Hence using (19) one gets the result

$$\max_{1\le k\le K2^{2m}}|V_{m,k}| \le 5\left|H - \frac{1}{2}\right|(1-H)^{-\frac{1}{2}}(C\log_* K)^{\frac{1}{2}}\,m^{\frac{1}{2}}2^{-2Hm}, \qquad (22)$$

with the exception of a set of probability at most $2\left(K2^{2m}\right)^{1-C}$, where $m \ge 1$, $K > 0$ and $C > 1$ are arbitrary.

(d) The maximum of $U_{m,k}$. We divide the half line into intervals of length L, where $L \ge 4K$. For definiteness, choose $L = 4K$. Apart from this, this part will be similar to part (b). In the sequel we use the convention that when the lower limit of a summation is a real number x, the summation starts at $\lceil x \rceil$, and similarly, if the upper limit is y, the summation ends at $\lfloor y \rfloor$. By (17),

$$|U_{m,k}| \le \sum_{j=1}^{\infty}\sum_{(j-1)L<t_v\le jL}\left|(t_k+t_{v+1})^{H-\frac{1}{2}}-(t_k+t_v)^{H-\frac{1}{2}}-t_{v+1}^{H-\frac{1}{2}}+t_v^{H-\frac{1}{2}}\right|$$

$$\times |B_{m+1}(-t_v)-B_m(-t_v)|$$

$$\le \sum_{j=1}^{\infty}\max_{(j-1)L<t_v\le jL}|B_{m+1}(-t_v)-B_m(-t_v)|\,(\Delta t)^{H-\frac{1}{2}}$$

$$\times \sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}}\left|(k+v+1)^{H-\frac{1}{2}}-(k+v)^{H-\frac{1}{2}}-(v+1)^{H-\frac{1}{2}}+v^{H-\frac{1}{2}}\right|. \quad (23)$$

Lemma 3 gives an upper bound for the maximal difference between two consecutive approximations of BM if $j \ge 1$ is an arbitrary fixed value:

$$\max_{(j-1)L<t_v\le jL}|B_{m+1}(-t_v)-B_m(-t_v)| \qquad (24)$$

$$\le (jL)^{\frac{1}{4}}\left(\log_*(jL)\right)^{\frac{3}{4}}m2^{-\frac{m}{2}}$$

$$\le \begin{cases} L^{\frac{1}{4}}(\log_* L)^{\frac{3}{4}}m2^{-\frac{m}{2}} & \text{if } j=1, \\ 2j^{\frac{1}{4}}(\log_* j)^{\frac{3}{4}}L^{\frac{1}{4}}(\log_* L)^{\frac{3}{4}}m2^{-\frac{m}{2}} & \text{if } j\ge 2, \end{cases} \qquad (25)$$

with the exception of a set of probability at most $3\left(jL2^{2m}\right)^{1-C}$, where $C > 1$ is arbitrary and $m \ge m_1(C)$. This implies for any $C \ge 3$ and $m \ge m_1(C)$ that the above inequality (25) holds simultaneously for all $j = 1,2,3,\dots$ with the exception of a set of probability at most

$$3\left(L2^{2m}\right)^{1-C}\sum_{j=1}^{\infty}j^{1-C} < 3\left(L2^{2m}\right)^{1-C}\frac{\pi^2}{6} < \left(K2^{2m}\right)^{1-C}. \qquad (26)$$

15

For the other major factor in (23) binomial series are applied as above, with $m \geq 0$, $k \geq 1$, and $v \geq 1$:

$$
\begin{aligned}
A &= (k+v+1)^{H-\frac{1}{2}} - (k+v)^{H-\frac{1}{2}} = (k+v)^{H-\frac{1}{2}} \left[\left(1 + \frac{1}{k+v} \right)^{H-\frac{1}{2}} - 1 \right] \\
&= (k+v)^{H-\frac{1}{2}} \sum_{s=1}^{\infty} \binom{H - \frac{1}{2}}{s} \frac{1}{(k+v)^s},
\end{aligned}
$$

and for $v \geq 2$:

$$
B = (v+1)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}} = v^{H-\frac{1}{2}} \left[\left(1 + \frac{1}{v} \right)^{H-\frac{1}{2}} - 1 \right] = v^{H-\frac{1}{2}} \sum_{s=1}^{\infty} \binom{H - \frac{1}{2}}{s} \frac{1}{v^s},
$$

while $B = 2^{H-\frac{1}{2}} - 1$ when $v = 1$. Then A and B have the same sign, $0 \leq A_1 \leq |A| \leq |B| \leq B_2$, and so $|A - B| \leq B_2 - A_1$, where

$$
A_1 = (k+v)^{H-\frac{3}{2}} \left| H - \frac{1}{2} \right| - (k+v)^{H-\frac{5}{2}} \frac{1}{2} \left| H - \frac{1}{2} \right| \left(\frac{3}{2} - H \right)
$$

and $B_2 = v^{H-\frac{3}{2}} \left| H - \frac{1}{2} \right|$.

Thus if the second major factor in (23) is denoted by $C_{m,k,j}$, we obtain for any $j \geq 1$ that

$$
C_{m,k,j}
$$

$$
= (\Delta t)^{H-\frac{1}{2}} \sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} \left| (k+v+1)^{H-\frac{1}{2}} - (k+v)^{H-\frac{1}{2}} - (v+1)^{H-\frac{1}{2}} + v^{H-\frac{1}{2}} \right|
$$

$$
\leq \left| H - \frac{1}{2} \right| (\Delta t)^{H-\frac{1}{2}}
$$

$$
\times \sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} \left\{ v^{H-\frac{3}{2}} - (k+v)^{H-\frac{3}{2}} + \frac{1}{2} \left(\frac{3}{2} - H \right) (k+v)^{H-\frac{5}{2}} \right\}.
$$

For $H \neq \frac{1}{2}$ one can get the estimates for $j = 1$:

$$
\sum_{v=1}^{L2^{2m}} v^{H-\frac{3}{2}} < 1 + \int_1^{L2^{2m}} x^{H-\frac{3}{2}} \, \mathrm{d}x = \frac{(\Delta t)^{\frac{1}{2}-H}}{H-\frac{1}{2}} L^{H-\frac{1}{2}} + \frac{H-\frac{3}{2}}{H-\frac{1}{2}},
$$

and for $j \geq 2$:

$$
\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} v^{H-\frac{3}{2}} < \int_{(j-1)L2^{2m}}^{jL2^{2m}} x^{H-\frac{3}{2}} \, \mathrm{d}x
$$

$$
= \frac{(\Delta t)^{\frac{1}{2}-H}}{H-\frac{1}{2}} \left[(jL)^{H-\frac{1}{2}} - ((j-1)L)^{H-\frac{1}{2}} \right],
$$

further, for any $j \geq 1$,

$$\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} (k+v)^{H-\frac{3}{2}} > \int_{(j-1)L2^{2m}+1}^{jL2^{2m}+1} (k+x)^{H-\frac{3}{2}} \, \mathrm{d}x$$

$$= \frac{(\Delta t)^{\frac{1}{2}-H}}{H-\frac{1}{2}} \left[(t_{k+1}+jL)^{H-\frac{1}{2}} - (t_{k+1}+(j-1)L)^{H-\frac{1}{2}} \right],$$

and also for any $j \geq 1$,

$$\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} (k+v)^{H-\frac{5}{2}} < \int_{(j-1)L2^{2m}}^{jL2^{2m}} (k+x)^{H-\frac{5}{2}} \, \mathrm{d}x$$

$$= \frac{(\Delta t)^{\frac{3}{2}-H}}{\frac{3}{2}-H} \left[(t_k+(j-1)L)^{H-\frac{3}{2}} - (t_k+jL)^{H-\frac{3}{2}} \right].$$

Denote the sign of a real number x by ϵ_x (0 if $x = 0$). When $j = 1$, it follows that

$$C_{m,k,1} \leq \epsilon_{H-\frac{1}{2}} \left\{ L^{H-\frac{1}{2}} \left[1 - \left(1 + \frac{t_{k+1}}{L}\right)^{H-\frac{1}{2}} + \left(1 + \frac{t_{k+1}-L}{L}\right)^{H-\frac{1}{2}} \right] \right.$$

$$\left. + \left(H - \frac{3}{2}\right) (\Delta t)^{H-\frac{1}{2}} \right\}$$

$$+ \left|H - \frac{1}{2}\right| \frac{\Delta t}{2} L^{H-\frac{3}{2}} \left[\left(1 + \frac{t_k-L}{L}\right)^{H-\frac{3}{2}} - \left(1 + \frac{t_k}{L}\right)^{H-\frac{3}{2}} \right],$$

and similarly, when $j \geq 2$,

$$C_{m,k,j} \leq \epsilon_{H-\frac{1}{2}} \left[(jL)^{H-\frac{1}{2}} - ((j-1)L)^{H-\frac{1}{2}} \right.$$

$$\left. - (t_{k+1}+jL)^{H-\frac{1}{2}} + (t_{k+1}+(j-1)L)^{H-\frac{1}{2}} \right]$$

$$+ \left|H - \frac{1}{2}\right| \frac{\Delta t}{2} \left[(t_k+(j-1)L)^{H-\frac{3}{2}} - (t_k+jL)^{H-\frac{3}{2}} \right]$$

$$= \epsilon_{H-\frac{1}{2}} (jL)^{H-\frac{1}{2}}$$

$$\times \left[1 - \left(1 - \frac{1}{j}\right)^{H-\frac{1}{2}} - \left(1 + \frac{t_{k+1}}{jL}\right)^{H-\frac{1}{2}} + \left(1 + \frac{t_{k+1}-L}{jL}\right)^{H-\frac{1}{2}} \right]$$

$$+ \left|H - \frac{1}{2}\right| \frac{\Delta t}{2} (jL)^{H-\frac{3}{2}} \left[\left(1 + \frac{t_k-L}{jL}\right)^{H-\frac{3}{2}} - \left(1 + \frac{t_k}{jL}\right)^{H-\frac{3}{2}} \right],$$

Applying binomial series here again, first we get when $j \geq 2$ that

$$\epsilon_{H-\frac{1}{2}} \left[1 - \left(1 - \frac{1}{j}\right)^{H-\frac{1}{2}} \right] = \epsilon_{H-\frac{1}{2}} \sum_{s=1}^{\infty} \binom{H-\frac{1}{2}}{s} \frac{(-1)^{s+1}}{j^s}$$

$$\leq \left|H - \frac{1}{2}\right| \frac{1}{j} + \frac{1}{2} \left|H - \frac{1}{2}\right| \left(\frac{3}{2} - H\right) \frac{1}{j^2} \left(1 - \frac{1}{j}\right)^{-1}$$

$$\leq \left|H - \frac{1}{2}\right| \frac{1}{j} + \left|H - \frac{1}{2}\right| \left(\frac{3}{2} - H\right) \frac{1}{j^2},$$

17

since each term of the series is positive. Furthermore, with any $j \geq 1$,

$$\epsilon_{H-\frac{1}{2}}\left[\left(1 - \frac{L - t_{k+1}}{jL}\right)^{H-\frac{1}{2}} - \left(1 + \frac{t_{k+1}}{jL}\right)^{H-\frac{1}{2}}\right]$$

$$= \epsilon_{H-\frac{1}{2}}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{(-1)^s}{(jL)^s}\left[(L - t_{k+1})^s - (-t_{k+1})^s\right] \leq -\left|H - \frac{1}{2}\right|\frac{1}{j},$$

since each term of the series is negative: $L = 4K \geq 2t_{k+1}$, and the term in brackets is not larger than $2(L - \Delta t)^s$. Finally,

$$\left(1 - \frac{L - t_k}{jL}\right)^{H-\frac{3}{2}} - \left(1 + \frac{t_k}{jL}\right)^{H-\frac{3}{2}}$$

$$= \sum_{s=1}^{\infty}\binom{H-\frac{3}{2}}{s}\frac{(-1)^s}{(jL)^s}\left[(L - t_k)^s - (-t_k)^s\right]$$

$$\leq \left(\frac{3}{2} - H\right)\frac{4}{3j}\left(1 - \frac{L - \Delta t}{jL}\right)^{-1} \leq \left(\frac{3}{2} - H\right)\frac{4L}{3j\Delta t},$$

since each term of the series is positive and the term in brackets is not larger than $\frac{4}{3}(L - \Delta t)^s$.

Thus when $j \geq 2$ it follows for any $m \geq 0$, $k \geq 1$ that

$$C_{m,k,j}$$

$$\leq (jL)^{H-\frac{1}{2}}\left|H - \frac{1}{2}\right|\left(\frac{3}{2} - H\right)\frac{1}{j^2} + \left|H - \frac{1}{2}\right|\frac{\Delta t}{2}(jL)^{H-\frac{3}{2}}\left(\frac{3}{2} - H\right)\frac{4L}{3j\Delta t}$$

$$\leq a_H\left|H - \frac{1}{2}\right|L^{H-\frac{1}{2}}j^{H-\frac{5}{2}} \qquad \text{where } a_H = \begin{cases} \frac{5}{2} & \text{if } 0 < H < \frac{1}{2}, \\ \frac{5}{3} & \text{if } \frac{1}{2} < H < 1. \end{cases}$$

In a similar manner, when $j = 1$ one can get for any $m \geq 0$, $k \geq 1$ that

$$C_{m,k,1} \leq \epsilon_{H-\frac{1}{2}}L^{H-\frac{1}{2}} - \left|H - \frac{1}{2}\right|L^{H-\frac{1}{2}} + \epsilon_{H-\frac{1}{2}}\left(H - \frac{3}{2}\right)(\Delta t)^{H-\frac{1}{2}}$$

$$+ \left|H - \frac{1}{2}\right|\frac{\Delta t}{2}L^{H-\frac{3}{2}}\left(\frac{3}{2} - H\right)\frac{4L}{3\Delta t}$$

$$= \epsilon_{H-\frac{1}{2}}\left(\frac{3}{2} - H\right)\left[\frac{2}{3}HL^{H-\frac{1}{2}} - (\Delta t)^{H-\frac{1}{2}}\right]$$

$$\leq \begin{cases} \frac{3}{2}(\Delta t)^{H-\frac{1}{2}} & \text{if } 0 < H < \frac{1}{2}, \\ \frac{3}{8}L^{H-\frac{1}{2}} & \text{if } \frac{1}{2} < H < 1. \end{cases}$$

Then combine these results with (25) and (26) in (23). Using

$$\sum_{j=2}^{\infty}j^{\frac{1}{4}}(\log_* j)^{\frac{3}{4}}j^{H-\frac{5}{2}} < \int_1^{\infty}x^{H-\frac{9}{4}}\log_* x\,dx$$

$$= \int_1^e x^{H-\frac{9}{4}}\,dx + \int_e^{\infty}x^{H-\frac{9}{4}}\log x\,dx$$

$$= \left(\frac{5}{4} - H\right)^{-1} + \left(\frac{5}{4} - H\right)^{-2}e^{H-\frac{5}{4}}$$

$$< \begin{cases} 2.5 & \text{if } 0 < H < \frac{1}{2}, \\ 16.5 & \text{if } \frac{1}{2} < H < 1, \end{cases} \tag{27}$$

18

one can get the result of part (d). Consider first the case $\frac{1}{2} < H < 1$:

$$\max_{1 \le k \le K2^{2m}} |U_{m,k}|$$

$$\le \quad L^{\frac{1}{4}} (\log_* L)^{\frac{3}{4}} m 2^{-\frac{m}{2}} \frac{3}{8} L^{H-\frac{1}{2}} + 33 \, L^{\frac{1}{4}} (\log_* L)^{\frac{3}{4}} m 2^{-\frac{m}{2}} \frac{5}{3} \left| H - \frac{1}{2} \right| L^{H-\frac{1}{2}}$$

$$\le \quad \left(3 + 312 \left| H - \frac{1}{2} \right| \right) K^{H-\frac{1}{4}} (\log_* K)^{\frac{3}{4}} m 2^{-\frac{m}{2}}, \tag{28}$$

for any $C \ge 3$ and $m \ge m_1(C)$ with the exception of a set of probability at most $\left(K2^{2m} \right)^{1-C}$. (Recall that $L = 4K$.)

In the second case when $0 < H < \frac{1}{2}$ the above method apparently gives convergence here (just like in part (b)) only when $\frac{1}{4} < H < \frac{1}{2}$:

$$\max_{1 \le k \le K2^{2m}} |U_{m,k}|$$

$$\le \quad L^{\frac{1}{4}} (\log_* L)^{\frac{3}{4}} m 2^{-\frac{m}{2}} \frac{3}{2} (\Delta t)^{H-\frac{1}{2}} + 5 \, L^{\frac{1}{4}} (\log_* L)^{\frac{3}{4}} m 2^{-\frac{m}{2}} \frac{5}{2} \left| H - \frac{1}{2} \right| L^{H-\frac{1}{2}}$$

$$\le \quad 5 \, K^{\frac{1}{4}} (\log_* K)^{\frac{3}{4}} m 2^{-2m(H-\frac{1}{4})} + 36 \left| H - \frac{1}{2} \right| K^{H-\frac{1}{4}} (\log_* K)^{\frac{3}{4}} m 2^{-\frac{m}{2}}, \tag{29}$$

for any $C \ge 3$ and $m \ge m_1(C)$ with the exception of a set of probability at most $\left(K2^{2m} \right)^{1-C}$.

Now one can combine the results of parts (a), (b), (c), and (d), see (19), (20), (21), (22), (28), (29), to obtain the statement of the lemma. Remember that the rate of convergence in parts (a) and (c) is faster than the one in parts (b) and (d). Particularly, observe that there is a factor m in (b) and (d) which has a counterpart $m^{\frac{1}{2}}$ in (a) and (c). Since in the statement of this lemma we simply replaced the faster converging factors by the slower converging ones, the constant multipliers in (a) and (c) can be ignored if m is large enough. □ □

It is simple to extend formula (8) of the mth approximation $B_m^{(H)}$ of fBM to real arguments t by linear interpolation, just like in the case of the mth approximation $B_m(t)$ of ordinary BM, see e.g. in [Szabados (1996)]. So let $m \ge 0$ and $k \ge 0$ be integers, $\gamma \in [0, 1]$, and define that

$$B_m^{(H)}(t_{k+\gamma}) = \gamma B_m^{(H)}(t_{k+1}) + (1-\gamma) B_m^{(H)}(t_k) \tag{30}$$

$$= \quad \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k} \left[(t_k - t_{r-1})^{H-\frac{1}{2}} - (t_k - t_r)^{H-\frac{1}{2}} \right] B_m(t_{r+\gamma})$$

$$+ \left[(-t_r)_+^{H-\frac{1}{2}} - (-t_{r-1})_+^{H-\frac{1}{2}} \right] B_m(t_r).$$

Then the resulting *continuous parameter approximations of fBM* $B_m^{(H)}(t)$ $(t \ge 0)$ have continuous, piecewise linear sample paths. With this definition we are ready to state a main result of this paper.

Theorem 2. *For any* $H \in (\frac{1}{4}, 1)$, *the sequence* $B_m^{(H)}(t)$ $(t \ge 0, m = 0, 1, 2, \dots)$ *a.s. uniformly converges to a fBM* $W^{(H)}(t)$ $(t \ge 0)$ *on any compact interval* $[0, K]$, $K > 0$. *If* $K > 0$, $C \ge 3$, *and* $m \ge m_4(C)$, *it follows that*

$$\mathbf{P}\left\{ \max_{0 \le t \le K} |W^{(H)}(t) - B_m^{(H)}(t)| \ge \frac{\alpha(H, K)}{\left(1 - 2^{-\beta(H)}\right)^2} m 2^{-\beta(H)m} \right\} \le 9(K2^{2m})^{1-C},$$

where $\alpha(H, K)$ and $\beta(H) = \min(2H - \frac{1}{2}, \frac{1}{2})$ are the same as in Lemma 4. (The case $H = \frac{1}{2}$ is described by Theorem 1.)

Proof. At first we consider the maximum of $|B_{m+1}^{(H)}(t) - B_m^{(H)}(t)|$ for real $t \in [0, K]$. Lemma 4 gives an upper bound D_m for their maximal difference at vertices with $t = t_k = k\Delta t$:

$$\max_{0 \leq t_k \leq K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| \leq D_m,$$

except for an event of probability at most $8(K2^{2m})^{1-C}$. Since both $B_{m+1}^{(H)}(t)$ and $B_m^{(H)}(t)$ have piecewise linear sample paths, their maximal difference must occur at vertices of the sample paths. Let M_m denote the maximal increase of $B_m^{(H)}$ between pairs of points t_k, t_{k+1} in $[0, K]$:

$$\max_{0 \leq t_k \leq K} |B_m^{(H)}(t_{k+1}) - B_m^{(H)}(t_k)| \leq M_m,$$

except for an event of probability at most $2(K2^{2m})^{1-C}$, cf. (32) below. A sample path of $B_{m+1}^{(H)}(t)$ makes four steps on any interval $[t_k, t_{k+1}]$. To compute its maximal deviation from D_m it is enough to estimate its change between the midpoint and an endpoint of such an interval, at two steps from both the left and right endpoints:

$$\max_{0 \leq t_k \leq K} |B_{m+1}^{(H)}(t_{k \pm \frac{1}{2}}) - B_{m+1}^{(H)}(t_k)| \leq 2M_{m+1},$$

except for an event of probability at most $2(K2^{2(m+1)})^{1-C}$. Hence

$$\max_{0 \leq t_k \leq K} |B_{m+1}^{(H)}(t_{k+\frac{1}{2}}) - B_m^{(H)}(t_{k+\frac{1}{2}})|$$
$$= \max_{0 \leq t_k \leq K} \left| B_{m+1}^{(H)}(t_{k+\frac{1}{2}}) - \frac{1}{2}\left(B_m^{(H)}(t_k) + B_m^{(H)}(t_{k+1})\right) \right|$$
$$\leq \max_{0 \leq t_k \leq K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| + \max_{0 \leq t_k \leq K} |B_{m+1}^{(H)}(t_{k \pm \frac{1}{2}}) - B_{m+1}^{(H)}(t_k)|$$
$$\leq D_m + 2M_{m+1},$$

except for an event of probability at most $(8 + 2^{3-2C})(K2^{2m})^{1-C}$. The explanation above shows that at the same time this gives the upper bound we were looking for:

$$\max_{0 \leq t \leq K} |B_{m+1}^{(H)}(t) - B_m^{(H)}(t)| \leq D_m + 2M_{m+1}, \tag{31}$$

except for an event of probability at most $(8 + 2^{3-2C})(K2^{2m})^{1-C}$.

Thus we have to find an upper estimate M_m. For that the large deviation inequality (12) will be used. By (7), the increment of $B_m^{(H)}(t)$ on $[t_k, t_{k+1}]$ is

$$A_{m,k} = |B_m^{(H)}(t_{k+1}) - B_m^{(H)}(t_k)|$$
$$= \frac{2^{-2Hm}}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{r=-\infty}^{k} \left[(k+1-r)^{H-\frac{1}{2}} - (k-r)^{H-\frac{1}{2}}\right] \tilde{X}_m(r+1).$$

Then a similar argument can be used as in the proof of Lemma 4, see e.g. part (a) there:

$$\frac{\Gamma^2\left(H+\frac{1}{2}\right)}{2^{-4Hm}}\mathbf{Var}(A_{m,k}) = \sum_{r=-\infty}^{k}\left[(k+1-r)^{H-\frac{1}{2}}-(k-r)^{H-\frac{1}{2}}\right]^2$$

$$= 1+\left(2^{H-\frac{1}{2}}-1\right)^2+\sum_{r=-\infty}^{k-2}(k-r)^{2H-1}\left[\left(1+\frac{1}{k-r}\right)^{H-\frac{1}{2}}-1\right]^2$$

$$\le 1+\left(2^{H-\frac{1}{2}}-1\right)^2+\sum_{r=-\infty}^{k-2}(k-r)^{2H-1}\left(H-\frac{1}{2}\right)^2\frac{1}{(k-r)^2}$$

$$\le 1+\left(H-\frac{1}{2}\right)^2+\left(H-\frac{1}{2}\right)^2\frac{1}{2-2H}\le\frac{5}{2}\left(H-\frac{1}{2}\right)^2(1-H)^{-1}.$$

Hence taking $N=K2^{2m}$ and $C>1$ in (12), and using (19) too, one obtains for $m\ge 1$ that

$$M_m = \max_{1\le k\le K2^{2m}}|A_{m,k}|$$

$$\le \frac{5/\sqrt{2}}{\Gamma\left(H+\frac{1}{2}\right)}\left|H-\frac{1}{2}\right|(1-H)^{-\frac{1}{2}}(C\log_* K)^{\frac{1}{2}}\,m^{\frac{1}{2}}2^{-2Hm}, \quad (32)$$

with the exception of a set of probability at most $2\left(K2^{2m}\right)^{1-C}$, where $K>0$ and $C>1$ are arbitrary.

Then substituting this and Lemma 4 into (31), it follows that when $K>0$, $C\ge 3$, and $m\ge m_4(C)$,

$$\max_{0\le t\le K}|B_{m+1}^{(H)}(t)-B_m^{(H)}(t)|\le\alpha(H,K)m2^{-\beta(H)m} \quad (33)$$

except for an event of probability at most $8.125(K2^{2m})^{1-C}$ where $\alpha(H,K)$ and $\beta(H)$ are the same as in Lemma 4. Remember that the rate of convergence in (32), just like in parts (a) and (c) of the proof of Lemma 4, is faster than the one in parts (b) and (d) of that proof. Apart from constant multipliers, the result of (32) has the same form as the results of (a) and (c) there. Since in the statement of this theorem we simply replaced the faster converging factors by the slower converging ones, the constant multipliers of (32) can be ignored if m is large enough. This is why the $\alpha(H,K)$ defined by Lemma 4 is suitable here too.

In the second part of the proof we compare $B_m^{(H)}(t)$ to $B_{m+j}^{(H)}(t)$, where $j\ge 1$ is an arbitrary integer. If $K>0$, $C\ge 3$, and $m\ge m_4(C)$, then (33) implies that

$$\max_{0\le t\le K}|B_{m+j}^{(H)}(t)-B_m^{(H)}(t)|\le\sum_{k=m}^{m+j-1}\max_{0\le t\le K}|B_{k+1}^{(H)}(t)-B_k^{(H)}(t)|$$

$$\le \sum_{k=m}^{\infty}\alpha(H,K)k2^{-\beta(H)k}\le\frac{\alpha(H,K)}{\left(1-2^{-\beta(H)}\right)^2}m2^{-\beta(H)m}.$$

21

Hence one can get that

$$\mathbf{P}\left\{\sup_{j\geq 1}\max_{0\leq t\leq K}|B^{(H)}_{m+j}(t) - B^{(H)}_m(t)| \geq \frac{\alpha(H,K)}{\left(1 - 2^{-\beta(H)}\right)^2}m2^{-\beta(H)m}\right\}$$

$$\leq \sum_{k=m}^{\infty} 8.125\left(K2^{2k}\right)^{1-C} \leq 9\left(K2^{2m}\right)^{1-C}. \quad (34)$$

By the Borel–Cantelli lemma this implies that with probability 1, the sample paths of $B^{(H)}_m(t)$ converge uniformly to a process $W^{(H)}(t)$ on any compact interval $[0,K]$. Then $W^{(H)}(t)$ has continuous sample paths, and inherits the properties of $B^{(H)}_m(t)$ described in Section 3: it is a centered, self-similar process with stationary increments. As Lemma 5 below implies, the process $\left(W^{(H)}(t) : t \geq 0\right)$ so defined is Gaussian. Therefore $W^{(H)}(t)$ is a fBM and by (34) the convergence rate of the approximation is the one stated in the theorem. □ □

The aim of the next lemma to show that integration by parts is essentially valid for (2) representing $W^{(H)}(t)$, resulting a formula similar to (10). Then it follows that $\left(W^{(H)}(t) : t \geq 0\right)$ can be stochastically arbitrarily well approximated by a linear transform of the Gaussian process $(W(t) : t \geq 0)$, so it is also Gaussian.

Lemma 5. *Let $W^{(H)}(t)$ be the process whose existence is proved in Theorem 2 above for $H \in (\frac{1}{4}, 1)$, or, by a modified construction, in Theorem 3 below for any $H \in (0, 1)$. Then for any $t > 0$ and $\epsilon > 0$ there exists a $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ we have*

$$\mathbf{P}\left\{\left|W^{(H)}(t) - W^{(H)}_\delta(t) - \frac{\delta^{H-\frac{1}{2}}W(t-\delta)}{\Gamma\left(H + \frac{1}{2}\right)}\right| \geq \epsilon\right\} \leq \epsilon, \quad (35)$$

where

$$W^{(H)}_\delta(t) := -\int_{[-\frac{1}{\delta}, -\delta] \cup [0, t-\delta]} h'_s(s,t)\,W(s)\,\mathrm{d}s, \quad (36)$$

and $h(s,t)$ is defined by (9). ($W^{(H)}_\delta(t)$ is almost surely well-defined pathwise as an integral of a continuous function.)

The lemma shows that as $\delta \to 0+$, $W^{(H)}_\delta(t)$ stochastically converges to $W^{(H)}(t)$ when $H > \frac{1}{2}$, while $W^{(H)}_\delta(t)$ has a singularity given by the extra term in (35) when $H < \frac{1}{2}$. (If $H = \frac{1}{2}$ then $W^{(H)}_\delta(t) = 0$ and the lemma becomes trivial.)

Proof. Fix $t > 0$ and $\epsilon > 0$ and take any δ, $0 < \delta \leq t$. Let us introduce the notation, cf. (8):

$$B^{(H)}_{m,\delta}(t_{(m)}) = \sum_{t_r \in I_{m,\delta}} \frac{h(t_r - \Delta t, t_{(m)}) - h(t_r, t_{(m)})}{\Delta t}B_m(t_r)\,\Delta t, \quad (37)$$

where $I_{m,\delta} = \left(\left(\frac{-1}{\delta}\right)_{(m)} < t_r \leq -\delta_{(m)}\right] \cup (0 < t_r \leq t_{(m)} - \delta_{(m)}]$ and the abbreviation $s_{(m)} = \lfloor s2^{2m}\rfloor 2^{-2m}$ is used for $s = t, \delta$, and $-1/\delta$ (an empty sum being

zero by convention. Then we get the inequality

$$
\left| W^{(H)}(t) - W_\delta^{(H)}(t) - \frac{\delta^{H-\frac{1}{2}} W(t-\delta)}{\Gamma\left(H+\frac{1}{2}\right)} \right| \le |W^{(H)}(t) - B_m^{(H)}(t_{(m)})|
$$

$$
+ \left| B_m^{(H)}(t_{(m)}) - B_{m,\delta}^{(H)}(t_{(m)}) - \frac{\delta_{(m)}^{H-\frac{1}{2}} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma\left(H+\frac{1}{2}\right)} \right|
$$

$$
+ |B_{m,\delta}^{(H)}(t_{(m)}) - W_\delta^{(H)}(t)|
$$

$$
+ \left| \frac{\delta_{(m)}^{H-\frac{1}{2}} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma\left(H+\frac{1}{2}\right)} - \frac{\delta^{H-\frac{1}{2}} W(t-\delta)}{\Gamma\left(H+\frac{1}{2}\right)} \right|. \tag{38}
$$

First we have to estimate the second term on the right hand side as $\delta \to 0+$, uniformly in m (this requires the longest computation):

$$
B_m^{(H)}(t_{(m)}) - B_{m,\delta}^{(H)}(t_{(m)}) - \frac{\delta_{(m)}^{H-\frac{1}{2}} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma\left(H+\frac{1}{2}\right)} =: E_{m,\delta} + F_{m,\delta} + G_{m,\delta},
$$

where

$$
E_{m,\delta} = \sum_{t_{(m)}-\delta_{(m)}<t_r\le t_{(m)}} \frac{h(t_r - \Delta t, t_{(m)}) - h(t_r, t_{(m)})}{\Delta t} B_m(t_r) \, \Delta t
$$

$$
- \frac{\delta_{(m)}^{H-\frac{1}{2}} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma\left(H+\frac{1}{2}\right)},
$$

$$
F_{m,\delta} = \sum_{-\delta_{(m)}<t_r\le 0} \frac{h(t_r - \Delta t, t_{(m)}) - h(t_r, t_{(m)})}{\Delta t} B_m(t_r) \, \Delta t
$$

and

$$
G_{m,\delta} = \sum_{-\infty<t_r\le \left(\frac{-1}{\delta}\right)_{(m)}} \frac{h(t_r - \Delta t, t_{(m)}) - h(t_r, t_{(m)})}{\Delta t} B_m(t_r) \, \Delta t.
$$

Then "summation by parts" shows that

$$
E_{m,\delta} = \sum_{t_{(m)}-\delta_{(m)}<t_r< t_{(m)}} h(t_r, t_{(m)})[B_m(t_{r+1}) - B_m(t_r)].
$$

(This is the point where the extra term in the definition of $E_{m,\delta}$ is needed.)
Thus

$$
\mathbf{Var}\left(\Gamma\left(H+\frac{1}{2}\right) E_{m,\delta}\right) = \sum_{t_{(m)}-\delta_{(m)}<t_r< t_{(m)}} (t_{(m)} - t_r)^{2H-1} 2^{-2m}
$$

$$
= t_{(m)}^{2H-1} \sum_{t_{(m)}-\delta_{(m)}<t_r< t_{(m)}} \left(1 - \frac{t_r}{t_{(m)}}\right)^{2H-1} \Delta t
$$

$$
\le t_{(m)}^{2H-1} \int_{t_{(m)}-\delta_{(m)}}^{t_{(m)}} \left(1 - \frac{u}{t_{(m)}}\right) \, du
$$

$$
= \frac{\delta_{(m)}^{2H}}{2H} \le \frac{\delta^{2H}}{2H},
$$

23

for any $m \geq 0$. Then by the large deviation inequality (11), for any $m \geq 0$ and for any $C > 0$,

$$\mathbf{P}\left\{|E_{m,\delta}| \geq \left(2C\log_*\left(\frac{1}{\delta}\right)\right)^{\frac{1}{2}} \frac{\delta^H}{\Gamma\left(H+\frac{1}{2}\right)(2H)^{\frac{1}{2}}}\right\} \leq 2\delta^C. \qquad (39)$$

Similarly as above, the definition of $F_{m,\delta}$ can be rewritten using "summation by parts" that gives

$$F_{m,\delta} = \sum_{-\delta_{(m)} < t_r < 0} h(t_r, t_{(m)})[B_m(t_{r+1}) - B_m(t_r)] + h(-\delta_{(m)}, t_{(m)})B_m(-\delta_{(m)} + \Delta t).$$

The definition of $F_{m,\delta}$ shows that it is equal to zero whenever $\delta < \Delta t$, therefore when giving an upper bound for its variance it can be assumed that $\delta \geq \Delta t$. Thus

$$\mathbf{Var}\left(\Gamma\left(H+\frac{1}{2}\right)F_{m,\delta}\right)$$

$$= \sum_{0 < t_v < \delta_{(m)}} \left[(t_{(m)} + t_v)^{H-\frac{1}{2}} - t_v^{H-\frac{1}{2}}\right]^2 \Delta t + \left[(t_{(m)} + \delta_{(m)})^{H-\frac{1}{2}} - \delta_{(m)}^{H-\frac{1}{2}}\right]^2$$

$$\times (\delta_{(m)} - \Delta t)$$

$$\leq t_{(m)}^{2H-1} \sum_{0 < t_v < \delta_{(m)}} \left[\left(1 + \frac{t_v}{t_{(m)}}\right)^{2H-1} + \left(\frac{t_v}{t_{(m)}}\right)^{2H-1}\right] \Delta t$$

$$+ \left[(t_{(m)} + \delta_{(m)})^{2H-1} + \delta_{(m)}^{2H-1}\right] \delta_{(m)}$$

$$\leq t_{(m)}^{2H-1} \int_0^{\delta_{(m)}} \left[\left(1 + \frac{u}{t_{(m)}}\right)^{2H-1} + \left(\frac{u}{t_{(m)}}\right)^{2H-1}\right] dt + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H}$$

$$= \frac{t_{(m)}^{2H}}{2H}\left[\left(1 + \frac{\delta_{(m)}}{t_{(m)}}\right)^{2H} + \left(\frac{\delta_{(m)}}{t_{(m)}}\right)^{2H} - 1\right] + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H}$$

$$\leq \frac{3}{2}t_{(m)}^{2H-1}\delta_{(m)} + \frac{\delta_{(m)}^{2H}}{2H} + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H} \leq \frac{7}{2}t^{2H-1}\delta + \frac{3}{2H}\delta^{2H}.$$

So by the large deviation inequality (11), for any $m \geq 0$ and for any $C > 0$,

$$\mathbf{P}\left\{|F_{m,\delta}| \geq \left(2C\log_*\left(\frac{1}{\delta}\right)\right)^{\frac{1}{2}} \frac{\left(\frac{7}{2}t^{2H-1}\delta + \frac{3}{2H}\delta^{2H}\right)^{\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}\right\} \leq 2\delta^C. \qquad (40)$$

Proceeding in a similar way with $G_{m,\delta}$, one obtains that

$$G_{m,\delta} = \sum_{-\infty < t_r < \left(\frac{-1}{\delta}\right)_{(m)}} h(t_r, t_{(m)})[B_m(t_{r+1}) - B_m(t_r)]$$

$$- h\left(\left(\frac{-1}{\delta}\right)_{(m)}, t_{(m)}\right) B_m\left(\left(\frac{-1}{\delta}\right)_{(m)}\right).$$

24

Hence

$$\mathbf{Var}(\Gamma\left(H+\frac{1}{2}\right)G_{m,\delta}) = \sum_{-\left(\frac{-1}{\delta}\right)_{(m)}<t_v<\infty}\left[(t_{(m)}+t_v)^{H-\frac{1}{2}}-t_v^{H-\frac{1}{2}}\right]^2\Delta t$$

$$+\left[\left(t_{(m)}-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)^{H-\frac{1}{2}}-\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)^{H-\frac{1}{2}}\right]^2\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)$$

$$\leq \sum_{-\left(\frac{-1}{\delta}\right)_{(m)}<t_v<\infty}t_v^{2H-1}\left(H-\frac{1}{2}\right)^2\left(\frac{t_{(m)}}{t_v}\right)^2\Delta t$$

$$+\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)^{2H-1}\left(H-\frac{1}{2}\right)^2\left(\frac{t_{(m)}}{-\left(\tfrac{-1}{\delta}\right)_{(m)}}\right)^2\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)$$

$$= \left(H-\frac{1}{2}\right)^2t_{(m)}^2\left\{\sum_{-\left(\frac{-1}{\delta}\right)_{(m)}<t_v<\infty}t_v^{2H-3}\Delta t+\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)^{2H-2}\right\}$$

$$\leq \left(H-\frac{1}{2}\right)^2t_{(m)}^2\left\{\int_{-\left(\frac{-1}{\delta}\right)_{(m)}}^{\infty}u^{2H-3}\,\mathrm{d}u+\left(-\left(\tfrac{-1}{\delta}\right)_{(m)}\right)^{2H-2}\right\}$$

$$\leq \frac{3(H-\frac{1}{2})^2}{2(1-H)}t^2\delta^{2-2H}.$$

So again by the large deviation inequality (11), for any $m\geq 0$ and for any $C>0$,

$$\mathbf{P}\left\{|G_{m,\delta}|\geq\left(2C\log_*\left(\frac{1}{\delta}\right)\right)^{\frac{1}{2}}\frac{|H-\frac{1}{2}|}{\Gamma\left(H+\frac{1}{2}\right)}\left(\frac{\frac{3}{2}}{1-H}\right)^{\frac{1}{2}}t\,\delta^{1-H}\right\}\leq 2\delta^C. \quad (41)$$

Combining (39), (40) and (41), it follows that there exists a $\delta_0>0$ such that for any $0<\delta<\delta_0$ and for any $m\geq 0$,

$$\mathbf{P}\left\{\left|B_m^{(H)}(t_{(m)})-B_{m,\delta}^{(H)}(t_{(m)})-\frac{\delta_{(m)}^{H-\frac{1}{2}}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma\left(H+\frac{1}{2}\right)}\right|\geq\frac{\epsilon}{4}\right\}\leq\frac{\epsilon}{4}.$$

After the second term on the right hand side of (38) we turn to the third term. Take now any $\delta\in(0,\delta_0)$ Since $h(s,t)$ has continuous partial derivative w.r.t. s on the intervals $[-1/\delta,-\delta]$ and $[\delta,t-\delta]$ and by Theorem 1, B_m a.s. uniformly converges to the Wiener process W on these intervals, comparing (36) and (37) shows that with this δ there exists an m such that

$$\mathbf{P}\left\{|B_{m,\delta}^{(H)}(t_{(m)})-W_{\delta}^{(H)}(t)|\geq\frac{\epsilon}{4}\right\}\leq\frac{\epsilon}{4}.$$

Theorem 1 also implies that m can be chosen so that for the fourth term in (38) one similarly has

$$\mathbf{P}\left\{\left|\frac{\delta_{(m)}^{H-\frac{1}{2}}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma\left(H+\frac{1}{2}\right)}-\frac{\delta^{H-\frac{1}{2}}W(t-\delta)}{\Gamma\left(H+\frac{1}{2}\right)}\right|\geq\frac{\epsilon}{4}\right\}\leq\frac{\epsilon}{4}.$$

25

Finally, Theorem 2 (or, with a modified construction, Theorem 3 below) guarantees that m can be chosen so that the first term in (38) satisfies the same inequality:

$$\mathbf{P}\left\{|W^{(H)}(t) - B_m^{(H)}(t)| \geq \frac{\epsilon}{4}\right\} \leq \frac{\epsilon}{4}.$$

The last four formulae together prove the lemma. $\qquad\square\qquad\qquad\square$

5 Improved construction using the KMT approximation

Parts (b) and (d) of the proof of Lemma 4 gave worse rate of convergence than parts (a) and (c), in which the rates can be conjectured to be best possible. The reason for this is clearly the relatively weaker convergence rate of the RW approximation of ordinary BM, that was used in parts (b) and (d), but not in parts (a) and (c). It is also clear from there that using the best possible KMT approximation instead would eliminate this weakness and would give hopefully the best possible rate here too. The price one has to pay for this is the intricate and "future-dependent" procedure by which the KMT method constructs suitable approximating RWs from BM.

The result we need from Komlós, Major, and Tusnády (1975, 1976) is as follows. Suppose that one wants to define an i.i.d. sequence $X_1, X_2, \ldots$ of random variables with a given distribution so that the partial sums are as close to BM as possible. Assume that $\mathbf{E}(X_k) = 0$, $\mathbf{Var}(X_k) = 1$ and the moment generating function $\mathbf{E}\left(e^{uX_k}\right) < \infty$ for $|u| \leq u_0$, $u_0 > 0$. Let $S(k) = X_1 + \cdots + X_k$, $k \geq 1$ be the partial sums. If BM $W(t)$ $(t \geq 0)$ is given, then for any $n \geq 1$ there exists a sequence of conditional quantile transformations applied to $W(1), W(2), \ldots, W(n)$ so that one obtains the desired partial sums $S(1), S(2), \ldots, S(n)$ and the difference between the two sequences is the smallest possible:

$$\mathbf{P}\left\{\max_{0 \leq k \leq n} |S(k) - W(k)| > C_0 \log n + x\right\} < K_0 e^{-\lambda x}, \qquad (42)$$

for any $x > 0$, where C_0, K_0, λ are positive constants that may depend on the distribution of X_k, but not on n or x. Moreover, λ can be made arbitrarily large by choosing a large enough C_0. Taking $x = C_0 \log n$ here one obtains

$$\mathbf{P}\left\{\max_{0 \leq k \leq n} |S(k) - W(k)| > 2C_0 \log n\right\} < K_0 n^{-\lambda C_0}, \qquad (43)$$

where $n \geq 1$ is arbitrary.

Fix an integer $m \geq 0$, and introduce the same notations as in previous sections: $\Delta t = 2^{-2m}$, $t_x = x\Delta t$. Then multiply the inner inequality in (43) by 2^{-m} and use self-similarity (1) of BM (with $H = \frac{1}{2}$) to obtain a shrunken RW $B_m^*(t_k) = 2^{-m}S_m(k)$ $(0 \leq k \leq K2^{2m})$ from the corresponding dyadic values $W(t_k)$ $(0 \leq k \leq K2^{2m})$ of BM by a sequence of conditional quantile transformations so that

$$\max_{0 \leq t_k \leq K} |B_m^*(t_k) - W(t_k)| \leq 2C_0 2^{-m} \log_*(K2^{2m}) \leq 5C_0 \log_* K \, m2^{-m}, \quad (44)$$

with the exception of a set of probability smaller than $K_0(K2^{2m})^{-\lambda C_0}$, for any $m \geq 1$ and $K > 0$. [Here (19) was used too.] Then (44) implies for the difference of two consecutive approximations that

$$\mathbf{P}\left\{\max_{0 \leq t_k \leq K} |B_{m+1}^*(t_k) - B_m^*(t_k)| > 10 C_0 \log_* K \, m2^{-m}\right\}$$
$$< 2K_0 \left(K2^{2m}\right)^{-\lambda C_0} \quad (45)$$

for any $m \geq 1$ and $K > 0$. This is exactly that we need to improve the rates of convergence in parts (b) and (d) of Lemma 4.

Substitute these KMT approximations $B_m^*(t_r)$ into definition (7) or (8) of $B_m^{(H)}(t_k)$. This way one can obtain faster converging approximations $B_m^{*(H)}$ of fBM. Then everything above in Sections 3 and 4 are still valid, except that one can use the improved formula (45) instead of Lemma 3 at parts (b) and (d) in the proof of Lemma 4. This way, instead of (21) one gets

$$\max_{1 \leq k \leq K2^{2m}} |Y_{m,k}| \leq \begin{cases} 23 C_0 \log_* K \, m2^{-2Hm} & \text{if } 0 < H < \frac{1}{2}, \\ 15 C_0 \log_* K \, K^{H-\frac{1}{2}} m2^{-m} & \text{if } \frac{1}{2} < H < 1, \end{cases} \quad (46)$$

for any $m \geq 1$, except for a set of probability smaller than $2K_0(K2^{2m})^{-\lambda C_0}$.

Also by (45), instead of (25) and (26) one has the improved inequalities:

$$\max_{(j-1)L < t_v \leq jL} \left|B_{m+1}^*(-t_v) - B_m^*(-t_v)\right|$$
$$\leq \begin{cases} 10 C_0 \log_* L \, m2^{-m} & \text{if } j = 1, \\ 14 C_0 \log_* j \, \log_* L \, m2^{-m} & \text{if } j \geq 2, \end{cases} \quad (47)$$

with the exception of a set of probability smaller than $2K_0(jL2^{2m})^{-\lambda C_0}$, where $m \geq 1$. If C_0 is chosen large enough so that $\lambda C_0 \geq 2$, then (47) holds simultaneously for all $j = 1, 2, 3, \ldots$ except for a set of probability smaller than

$$2K_0(L2^{2m})^{-\lambda C_0} \sum_{j=1}^{\infty} j^{-\lambda C_0} < \frac{1}{4} K_0(K2^{2m})^{-\lambda C_0}. \quad (48)$$

(Remember that we chose $L = 4K$ in part (d) of the proof of Lemma 4.) Then using this in part (d) of Lemma 4, instead of (27) one needs the estimate

$$\sum_{j=2}^{\infty} j^{H-\frac{5}{2}} \log_* j < \int_1^{\infty} x^{H-\frac{5}{2}} \log_* x \, \mathrm{d}x < 1 \qquad (0 < H < 1).$$

Then instead of (28) and (29), the improved results are as follows. First, in the case $\frac{1}{2} < H < 1$ one has

$$\max_{1 \leq k \leq K2^{2m}} |U_{m,k}|$$
$$\leq 10 C_0 \log_* L \, m2^{-m} \frac{3}{8} L^{H-\frac{1}{2}} + 14 C_0 \log_* L \, m2^{-m} \frac{5}{3} \left|H - \frac{1}{2}\right| L^{H-\frac{1}{2}}$$
$$\leq \left(18 + 112 \left|H - \frac{1}{2}\right|\right) C_0 \log_* K \, K^{H-\frac{1}{2}} m2^{-m} \quad (49)$$

27

for any $m \geq 1$ and C_0 large enough so that $\lambda C_0 \geq 2$, except for a set of probability smaller than given by (48).

Now in the case $0 < H < \frac{1}{2}$ it follows that

$$\max_{1 \leq k \leq K 2^{2m}} |U_{m,k}|$$

$$\leq \quad 10 C_0 \log_* L \, m 2^{-m} \frac{3}{2} (\Delta t)^{H - \frac{1}{2}} + 14 C_0 \log_* L \, m 2^{-m} \frac{5}{2} \left| H - \frac{1}{2} \right| L^{H - \frac{1}{2}}$$

$$\leq \quad C_0 \log_* K \, m \left(36 \cdot 2^{-2Hm} + 84 \left| H - \frac{1}{2} \right| K^{H - \frac{1}{2}} 2^{-m} \right) \qquad (50)$$

for any $m \geq 1$ and C_0 large enough so that $\lambda C_0 \geq 2$, except for a set of probability smaller than given by (48).

As a result, there is convergence for any $H \in (0,1)$. Since the KMT approximation itself has best possible rate for approximating ordinary BM by RW, it can be conjectured that the resulting convergence rates in the next lemma and theorem are also best possible (apart from constant multipliers) for approximating fBM by moving averages of a RW.

Lemma 6. *For any $H \in (0,1)$, $m \geq 1$, $K > 0$, $C > 1$, and C_0 large enough, we have*

$$\mathbf{P} \left\{ \max_{0 \leq t_k \leq K} |B_{m+1}^{*(H)}(t_k) - B_m^{*(H)}(t_k)| \geq \alpha^* m 2^{-\beta^*(H) m} \right\}$$

$$\leq \quad 4(K 2^{2m})^{1-C} + 3 K_0 (K 2^{2m})^{-\lambda C_0},$$

where $t_k = k 2^{-2m}$, $\beta^*(H) = \min(2H, 1)$, $\alpha^* = \alpha^*(H, K, C, C_0)$,

$$\alpha^* = \frac{(\log_* K)^{\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)} \left[10 C^{\frac{1}{2}} \frac{\left| H - \frac{1}{2} \right|}{(1-H)^{\frac{1}{2}}} + C_0 (\log_* K)^{\frac{1}{2}} \left(59 + 84 \left| H - \frac{1}{2} \right| K^{H - \frac{1}{2}} \right) \right]$$

if $H \in \left(0, \frac{1}{2}\right)$,

$$\alpha^* = \frac{(\log_* K)^{\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)} \left[10 C^{\frac{1}{2}} \frac{\left| H - \frac{1}{2} \right|}{(1-H)^{\frac{1}{2}}} + C_0 (\log_* K)^{\frac{1}{2}} \left(33 + 112 \left| H - \frac{1}{2} \right| \right) K^{H - \frac{1}{2}} \right]$$

if $H \in \left(\frac{1}{2}, 1\right)$, and the constants λ, C_0 and K_0 are defined by the KMT approximation (42) with C_0 is chosen so large that $\lambda C_0 \geq 2$. [The case $H = \frac{1}{2}$ is described by (45).]

Proof. Combine the results of parts (a) and (c) in the proof of Lemma 4 and the improved inequalities above, that is, apply (19), (20), (46), (22), (49), and (50). Here too, we simply replace the faster converging factors by the slower converging ones, but the constant multipliers of faster converging terms cannot be ignored, since the lemma is stated for any $m \geq 1$. $\qquad \square \qquad \square$

Now we can extend the improved approximations of fBM to real arguments by linear interpolation, in the same way as we did with the original approximations, see (30). This way we get continuous parameter approximations $B_m^{*(H)}(t)$, $(t \geq 0)$ for $m = 0, 1, 2, \ldots$, with continuous, piecewise linear sample paths. Now we can state the second main result of this paper.

Theorem 3. *For any $H \in (0, 1)$, the sequence $B_m^{*(H)}(t)$ $(t \geq 0, m = 0, 1, 2, \dots)$ a.s. uniformly converges to a fBM $W^{(H)}(t)$ $(t \geq 0)$ on any compact interval $[0, K]$, $K > 0$. If $m \geq 1$, $K > 0$, $C \geq 2$, and C_0 is large enough, it follows that*

$$\mathbf{P}\left\{ \max_{0 \leq t \leq K} |W^{(H)}(t) - B_m^{*(H)}(t)| \geq \frac{\bar{\alpha}^*}{\left(1 - 2^{-\beta^*(H)}\right)^2} m 2^{-\beta^*(H)m} \right\}$$
$$\leq 6(K2^{2m})^{1-C} + 4K_0(K2^{2m})^{-\lambda C_0}$$

with

$$\bar{\alpha}^* = \alpha^* + \frac{10}{\Gamma\left(H + \frac{1}{2}\right)} C^{\frac{1}{2}} (\log_* K)^{\frac{1}{2}} \left| H - \frac{1}{2} \right| (1 - H)^{-\frac{1}{2}},$$

where α^* and $\beta^*(H)$ are the same as in Lemma 6. (In other words, in the definition of α^* in Lemma 6 the constant multiplier 10 has to be changed to 20 here.) The constants λ, C_0, K_0 are defined by the KMT approximation (42) with C_0 is chosen so large that $\lambda C_0 \geq 2$. [The case $H = \frac{1}{2}$ is described by (44).]

Proof. The proof can follow the line of the proof of Theorem 2 with one exception: the constant multipliers in (32) and consequently in (31) cannot be ignored here. This is why the multiplier α^* of Lemma 6 had to be modified in the statement of the theorem. □ □

It can be conjectured that the best rate of approximation of fBM by moving averages of simple RW's is $O(N^{-H} \log N)$, where N is the number of points considered. Though it seems quite possible that definition of $B_m^{*(H)}(t)$ above, see (7) with the KMT approximations $B_m^*(t_r)$, supplies this rate of convergence for any $H \in (0, 1)$, but in Theorem 3 we were able to prove this rate only when $H \in (0, \frac{1}{2})$. A possible explanation could be that in parts (b) and (d) of Lemma 4 we separated the maxima of the kernel and the "integrator" parts.

As a result, the convergence rate we were able to prove when $\frac{1}{2} < H < 1$ is the same $O(N^{-\frac{1}{2}} \log N)$ that the original KMT approximation (44) gives for ordinary BM, where $N = K2^{2m}$, though in this case the sample paths of fBM are smoother than that of BM. (See e.g. [Decreusefond and Üstünel (1998)].) On the other hand, the obtained convergence rate is worse than this, but still thought to be the best possible, $O(N^{-H} \log N)$, when $0 < H < \frac{1}{2}$, which heuristically can be explained by the more zigzagged sample paths of fBM in this case.

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