An Alternative Method for Solving a Certain Class of Fractional Kinetic Equations

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Abstract. An alternative method for solving the fractional kinetic equations solved earlier by Haubold and Mathai (2000) and Saxena et al. (2002, 2004a, 2004b) is recently given by Saxena and Kalla (2007). This method can also be applied in solving more general fractional kinetic equations than the ones solved by the aforesaid authors. In view of the usefulness and importance of the kinetic equation in certain physical problems governing reaction-diffusion in complex systems and anomalous diffusion, the authors present an alternative simple method for deriving the solution of the generalized forms of the fractional kinetic equations solved by the aforesaid authors and Nonnenmacher and Metzler (1995). The method depends on the use of the Riemann-Liouville fractional calculus operators. It has been shown by the application of Riemann-Liouville fractional integral operator and its interesting properties, that the solution of the given fractional kinetic equation can be obtained in a straight-forward manner. This method does not make use of the Laplace transform.

1 Introduction

The paper deals with the essential problem related to applications of Mittag-Leffler function and Riemann-Liouville fractional calculus operators to fractional order kinetic equations arising in modeling physical phenomena, governing diffusion in porous media and relaxation processes. As such it reveals the important role of these tools in applications of fractional calculus. The results are interesting and useful for wide range of applied scientists dealing with fractional order differential and fractional order integral equations. In a series of papers the authors have demonstrated the use of integral transforms in the solution of certain fractional kinetic equations (2002, 2004a 2004b), reaction-diffusion equations (2006a, 2006b), and fractional differential equations governing non-linear waves (2006c, 2006d). In the present paper it is shown by the application of Riemann-Liouville fractional calculus operators and its interesting properties that the given fractional kinetic equations can be easily solved. Fractional kinetic equations are studied by Zaslavsky (1994), Saichev and Zaslavsky (1997), Gloeckle and Nonnenmacher (1991), and Saxena, Mathai and Haubold (2002, 2004a, 2004b) due to their importance in the solution of certain applied problems governing reaction and relaxation in complex systems and anomalous diffusion. The use of fractional kinetic equations in many problems arising in science and engineering can be found in the monographs by Podlubny (1999), Hilfer (2000), and Kilbas, Srivastava and Trujillo (2006) and the various papers given therein. The Mittag-Leffler functions naturally occur as a solution of fractional order differential equation or a fractional order integral equation. Mittag-Leffler (1903) defined this function, known as Mittag-Leffler function in the literature, in terms of the power series

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}; \quad (\alpha \in C, Re(\alpha) > 0). \tag{1}$$

This function is generalized by Wiman (1905) in the form

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0). \tag{2}$$

According to Dzherbashyan (1966, p.118), both the functions defined by the equations (1) and (2) are entire functions of order $\rho = 1/\alpha$ and type $\sigma = 1$. A comprehensive detailed account of these functions is available from the monographs of Erdélyi, Magnus, Oberhettinger and Tricomi (1995, Chapter 18) and Dzherbashyan (1966, Chapter 2). The Riemann-Liouville operators of fractional calculus are defined in the books by Miller and Ross (1993), Oldham and Spanier (1974), Podlubny (1999) and Kilbas, Srivastava and Trujillo (2006) as

$$_\alpha D_t^{-\nu} N(t) := \frac{1}{\Gamma(\nu)} \int_\alpha^t (t-u)^{\nu-1}\, N(u)du, Re(\nu) > 0, t > a \tag{3}$$

with $_aD_t^0 N(t) = N(t)$, and

$$_aD_t^\mu N(t) := \frac{d^n}{dt^n}(_aD_t^{\mu-n} N(t)), Re(\mu) > 0, n-\mu > 0. \tag{4}$$

By virtue of the definitions (3), it is not difficult to show that

$$_aD_t^{-\nu}(t-a)^{\rho-1} = \frac{\gamma(\rho)}{\Gamma(\rho+\nu)}(t-a)^{\rho+\nu-1}, \tag{5}$$

where $Re(\nu) > 0, Re(\rho) > 0; t > a$. Also from (Podlubny, 1999, p.72, eq.(2.117)), we have

$$ {}_aD_t^{\nu}(t-a)^{\rho-1} = \frac{\gamma(\rho)}{\Gamma(\rho-\nu)}(t-a)^{\rho-\nu-1}, \tag{6} $$

where $Re(\nu) > 0, Re(\rho) > 0, t > a$. When $\rho = 1$ (6) reduces to an interesting formula

$$ {}_aD_t^{-\nu}1 = \frac{1}{\Gamma(1-\nu)}(t-a)^{-\nu}, t > a; \nu \neq 1, 2 \ldots \tag{7} $$

which is a remarkable result in the theory of fractional calculus and indicates that the fractional derivative of a constant is not zero.

We now proceed to derive and solve the fractional kinetic equations in the next section.

2 Derivation of the fractional kinetic equation and its solution

If we integrate the standard kinetic equation

$$ \frac{d}{dt}N_i(t) = -c_iN_i(t), (c_i > 0) \tag{8} $$

we obtain (Haubold and Mathai, 2000, p.58)

$$ N(t) - N(a) = -c_i \ {}_aD_t^{-1}N_i(t), \tag{9} $$

where ${}_aD_t^{-1}$ is the standard Riemann integral operator. Here we recall that in the original paper of Haubold and Mathai (2000), the number density of species, $N_i = N_i(t)$ is a function of time. Further we assume that $N_i(t = a) = N_a$ is the number density of species i at time $t = a$. If we drop the index i in (9) and generalize it, we arrive at the fractional kinetic equation

$$ N(t) - N_a = -c^{\nu} \ {}_aD_t^{-\nu}N(t) \tag{10} $$

Solution of (10). If we multiply both sides of (10) by $(-c^{\nu})^m \ {}_aD_t^{-m\nu)}$, we obtain

$$ (-c^{\nu})^m \ {}_aD_t^{-m\nu}N(t) - (-c^{\nu})(-c^{\nu})^m \ {}_aD_t^{-m\nu-\nu} \ N(t) = (-c^{\nu})^m \ {}_aD_t^{-m\nu}N_a. \tag{11} $$

Now summing up both sides of (2.4) for m from 0 to ∞, it yields

$$ \begin{aligned} &\sum_{m=0}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu}N(t) - \sum_{m=0}^{\infty}(-c^{\nu})^{m+1} \ {}_aD_t^{-m\nu-\nu}N(t) \\ = \ & N_a\sum_{m=0}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu}1, \end{aligned} \tag{12} $$

which on using the formula (5) yields

$$\begin{aligned} N(t) &= N_a \sum_{m=0}^{\infty} (-c^{\nu})^m [(t-a)^{m\nu}/\Gamma(m\nu+1)] && (13) \\ &= N_a E_{\nu}[-c^{\nu}(t-a)^{\nu}], t > a && (14) \end{aligned}$$

Thus we arrive at the following theorem:

Theorem 1. If $Re(\nu) > 0, Re(c) > 0$ then there exists the unique solution of the integral equation

$$N(t) - N_a = -c^{\nu}\ {}_aD_t^{-\nu}(t), \tag{15}$$

given by

$$N(t) = N_a E_{\nu}(-c^{\nu}(t-a)^{\nu}),\ t > a \tag{16}$$

with the Mittag-Leffler function defined by (1).

When $a \to 0$, (16) reduces to the following result given by Haubold and Mathai (2000, p.63):

Corollary 1.1. If, $Re(c) > 0$ then the unique solution of the integral equation

$$N(t) - N_0 = -c^{\nu}\ {}_0D_t^{-\nu}(t), \tag{17}$$

is given by

$$N(t) = N_0 E_{\nu}(-c^{\nu} t^{\nu}). \tag{18}$$

Remark 1. If we apply the operator ${}_aD_t^{\nu}$ from the left to (10) and make use of (7), we obtain the fractional differential equation

$${}_aD_t^{\nu} N(t) - N_a \frac{(t-a)^{-\nu}}{\Gamma(1-\nu)} = -c^{\nu} N(t),\ \ t > a \tag{19}$$

whose solution is also given by (16). When a tends to zero in (16), it reduces to one obtained by Nonnenmacher and Metzler (1995, p.156) for the fractional relaxation equation with c replaced by $1/c$.

Remark 2. The method adopted in deriving the solution of fractional kinetic equation (8) is similar to that used by Al-Saqabi and Tuan (1996) for solving differ integral equations.

3 Theorem 2.

If $min\,\{Re(\nu), Re(\mu)\} > 0, Re(c) > 0$, then there exists the unique solution of the integral equation

$$`N(t) - N_a t^{\mu-1} = -c^{\nu}\ {}_aD_t^{-\nu}(t), \tag{20}$$

given by

$$N(t) = N_a\Gamma(\mu)(t-a)^{\mu-1}\ E_{\nu,\mu}(-c^\nu(t-a)^\nu), t > a \tag{21}$$

where $E_{\nu,\mu}(t)$is the generalized Mittag-Leffler function defined by (2).

Solution of (20). If we multiply both sides of (20) by $(-c^\nu)^m\ \ {}_aD_t^{-m\nu}$, we obtain

$$(-c^\nu)^m\ {}_aD_t^{-m\nu}N(t) - (-c^\nu)(-c^\nu)^m\ {}_aD_t^{-m\nu-\nu}N(t) = N_a(-c^\nu)^m\ {}_aD_t^{-m\nu}t^{\mu-1}. \tag{22}$$

Now summing up both sides of (22) for m from 0 to ∞, it yields

$$\sum_{m=0}^{\infty}(-c^\nu)^m\ {}_aD_t^{-m\nu}N(t) - \sum_{m=0}^{\infty}(-c^\nu)^{m+1}\ {}_aD_t^{-m\nu-\nu}N(t) = N_a\sum_{m=0}^{\infty}(-c^\nu)^m\ {}_aD_t^{-m\nu}t^{\mu-1}, \tag{23}$$

which on using the formula (5) gives

$$\begin{aligned} N(t) &= N_a\Gamma(\mu)\sum_{m=0}^{\infty}(-c^\nu)^m[(t-a)^{m\nu}/\gamma(m\nu+\mu)] && (24)\\ &= N_aE_{\nu,\mu}[-c^\nu(t-a)^\nu],\ t > a. && (25) \end{aligned}$$

This completes the proof of Theorem 2.

For $a = 0$, (25) reduces to the following result given by Saxena, Mathai and Haubold (2002, p.283).

Corollary 2.1. If min $\{Re(\nu), Re(\mu)\} > 0, R(c) > 0$ then the solution of the integral equation

$$N(t) - N_0t^{\mu-1} = -c^\nu\ {}_0D_t^{-\nu}(t) \tag{26}$$

‘ is given by

$$N(t) = N_0\Gamma(\mu)t^{\mu-1}\ E_{\nu,\mu}(-c^\nu t^\nu), \tag{27}$$

where $E_{\nu,\mu}(t)$is the generalized Mittag-Leffler function defined by (2).

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