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Fractional Poisson process

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Abstract

A fractional non-Markov Poisson stochastic process has been developed based on fractional generalization of the Kolmogorov–Feller equation. We have found the probability of n arrivals by time t for fractional stream of events. The fractional Poisson process captures long-memory effect which results in non-exponential waiting time distribution empirically observed in complex systems. In comparison with the standard Poisson process the developed model includes additional parameter μ. At $\mu = 1$ the fractional Poisson becomes the standard Poisson and we reproduce the well known results related to the standard Poisson process.

As an application of developed fractional stochastic model we have introduced and elaborated fractional compound Poisson process.

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1. Introduction

The main experimentally observed features of anomalous kinetic phenomena in complex systems are non-exponential time and non-Gaussian space patterns [1,2]. To describe the patterns fractional generalizations of the diffusion, diffusion–advection and Fokker–Planck type equations have been developed and studied recently [1–3]. From physical point of view the non-exponential evolution is caused by long-run memory effects in complex systems. From mathematical point of view fractional generalization of the kinetic equations results from substitution instead space and

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time derivatives the derivatives of fractional order. The current status and history of the fractional kinetic problem are presented in recent reviews [1,2].

One of important feature of statistical analysis of a counting random process is analysis of statistics of interarrival times. It is well known that the Poisson model predicts exponential probability distribution of interarrival times (see, for example [4]). Recently the empirical observations of failure of the Poisson model were found. The sizes (in number of bytes) or durations (measured in seconds) of a set of the Web network sessions or connections exhibit the long-tailed property, see, for example Fig. 5 in [5] and references there. In other words, the probability of duration of network sessions decreases by power law at large session-times instead of exponential decay predicted by the standard Poisson model and well confirmed empirically for phone communication connections.

To understand the origin of the observed power law asymptotic behavior of probability distribution function of interarrival times we propose non-Markov fractional Poisson model based on fractional generalization of the Kolmogorov–Feller equation [3,6,9].

To explain where the fractional Kolmogorov–Feller equation comes from let us remind the standard Kolmogorov–Feller equation for probability distribution function $P(x,t)$ [3,4]

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}y\, w(y)[P(x-y,t) - P(x,t)], \tag{1}$$

$$P(x, t=0) = \delta(x),$$

here $w(y)$ is probability density "to make a step" of the length y.

From point of view of the Montroll–Weiss continues time random walk (CTRW) model [1,6–9] the Kolmogorov–Feller equation (1) belongs to the type of master equations and describes situation when each consequent step of random length y is made after random waiting time t. Moreover randomness of step length is distributed in accordance with $w(y)$ while waiting time t has exponential distribution $\psi(t)$

$$\psi(t) = \mathrm{e}^{-t}, \quad t \geqslant 0. \tag{2}$$

Therefore, the exponential waiting time distribution $\psi(t)$ is the origin for the first order time derivative in the left side of Eq. (1). Fractional generalization of Eq. (1) is based on the fractional generalization of the waiting time distribution $\psi(t)$. In [3] the fractional waiting time distribution function $\psi_\mu(t)$ was proposed as the following integral (see Eq. (7.6) Ref. [3])

$$\psi_\mu(t) = \frac{\sin \pi\mu}{\pi} \int_0^{\infty} \frac{\mathrm{d}\rho\, \mathrm{e}^{-\rho t}}{\rho^{\mu} + \rho^{-\mu} + 2\cos \pi\mu}, \quad 0 < \mu \leqslant 1, \tag{3}$$

and called a fractional Poissonian distribution.

When the parameter $\mu \neq 1$ the fractional waiting time distribution becomes broader and possesses of non-exponential power-law behavior at large t. The spreading of the waiting time distribution leads to non-Markovian non-exponential evolution, the latter being a typical manifestation of temporal phenomena inherent in complex physical systems.

In this paper we have developed and elaborated the fractional Poisson distribution based on the fractional generalization of the Kolmogorov–Feller equation. We take the special form of $w(y)$, $w(y) = \nu\delta(y-1)$, where δ is delta function and the parameter ν has physical dimension $[\nu] = \mathrm{sec}^{-\mu}$

(see Eq. (19)). It results to probability distribution function of a counting process when the total number of "items" that have arrived up to time t is governed by the fractional stream. In comparison with the standard Poisson process the developed model includes additional parameter μ, $0 < \mu \leqslant 1$. Thus, the model provides fractional generalization of the standard Poisson process, to which it reduces at $\mu = 1$.

The paper is organized as follows.

In Section 1 we explain how and where the fractional Poisson comes from. The basic definitions of the standard Poisson random process are reminded in Section 2. In Section 3 we define the fractional Poisson model, obtain new probability distribution function $P_\mu(n,t)$, evaluate mean and variance, find probability distribution function of interarrival times. Fractional compound Poisson process is defined and elaborated in Section 4 as an application of the developed model. In Section 5 we discuss the relationships between the developed fractional model and the standard Poisson random process.

2. The Poisson process

2.1. Generation function

The standard Poisson process is concerned with the distribution of arrivals under applicable assumptions. The probability of a single arrival during a small time interval Δt is $\overline{n}\Delta t$, with rate $\overline{n}$ and more than a single arrival during Δt is negligible. Let $P(n,t)$ be the probability of n items having arrived by time t. The probability $P(n,t)$ satisfies the normalizing condition $\sum_{n=0}^{\infty} P(n,t) = 1$ because either nothing arrived or something must have arrived by time t. To see what happens during the subsequent small interval Δt we write

$$P(0,t+\Delta t) = P(0,t)(1-\overline{n}\Delta t), \tag{4}$$

$$P(n,t+\Delta t) = P(n,t)(1-\overline{n}\Delta t) + P(n-1,t)\overline{n}\Delta t, \quad n \geqslant 1. \tag{5}$$

The Eq. (4) gives the probability that no arrivals have occurred by time $t + \Delta t$. This probability may be related to the state of the system at time t. Thus, by the law of compound probabilities of two events which occur independently (i.e., one takes the product of the probabilities of these events), it is equal to the probability that nothing had arrived by the time t multiplied by the probability that nothing arrived during Δt. For the case $n \geqslant 1$ this property (i.e., having the same number at time t and nothing arriving during Δt) also holds, but in addition there might have been $n-1$ arrivals during time t, followed by an additional arrival during Δt. The product of these quantities yields the second term on the right of Eq. (5). We have not mentioned the possibility of more than one arrival during the small interval since it is negligible and can be shown to vanish in what follows.

On multiplying, transposing $P(n,t)$ to the left, and dividing by Δt, the Eq. (5) becomes

$$\frac{P(n,t+\Delta t) - P(n,t)}{\Delta t} = \overline{n}(P(n-1,t) - P(n,t)), \quad n \geqslant 1.$$

At the limit $\Delta t \to 0$ by definition, the left side is the first order time derivative and we have

$$\frac{\partial P(0,t)}{\partial t} = -\overline{n}P(0,t), \tag{6}$$

$$\frac{\partial P(n,t)}{\partial t} = \overline{n}(P(n-1,t) - P(n,t)), \quad n \geqslant 1, \tag{7}$$

$$P(n,t=0) = P_0(n).$$

Eq. (7) is linear differential equation with respect to t and difference equation with respect to n, generally called differential–difference equation.

To solve Eqs. (6) and (7) it is convenient to use the method of generating function. By introducing the generating function $G(s,t)$

$$G(s,t) = \sum_{n=0}^{\infty} s^n P(n,t), \tag{8}$$

we see from Eq. (8) that the probability $P(n,t)$ is obtained by differentiating $G(s,t)$ n times with respect to s, then dividing by $n!$, and putting $s=0$, that is

$$P(n,t) = \frac{1}{n!}\frac{\partial^n G(s,t)}{\partial s^n}\bigg|_{s=0}. \tag{9}$$

Now by multiplying Eq. (7) by s^n and summing over n as a result we obtain the following linear differential equation for the generating function $G(s,t)$:

$$\frac{\partial G(s,t)}{\partial t} = \overline{n}(s-1)G(s,t). \tag{10}$$

One should specify the initial at $t=0$ condition for the generating function. In general, the time origin $t=0$ for a specific study could be chosen anywhere, even after arrivals had actually occurred. Indeed, it may be that, by $t=0$, k units had arrived. In that case, $P(n,0) = P(n,t=0)$ is zero if $n \neq k$ and unity if $n=k$. Thus we can write

$$G(s,t=0) = \sum_{n=0}^{\infty} s^n P(n,t=0) = s^k P(k,t=0) = s^k. \tag{11}$$

It is easily to see that solution of the problem defined by Eqs. (10) and (11) has the form

$$G(s,t) = s^k \exp\{\overline{n}(s-1)t\}. \tag{12}$$

Further, suppose, as usual, that at $t=0$ nothing had arrived. Then $G(s,t=0)=1$ since $k=0$ and we get the equation for the generation function

$$G(s,t) = \exp\{\overline{n}(s-1)t\}. \tag{13}$$

Using Eqs. (13) and (9) we get the well known equation for the probability $P(n,t)$ of n items having arrived by time t

$$P(n,t) = \frac{(\overline{n}t)^n}{n!}e^{-\overline{n}t}. \tag{14}$$

This equation shows that the number of arrivals occurring in time interval t has a Poisson distribution with mean $\overline{n}t$. Hence, the physical meaning of the parameter $\overline{n}$ can be interpreted as the average number of arrivals occurring per unit time. The parameter $\overline{n}$ has physical dimension $[\overline{n}] = \sec^{-1}$.

2.2. Waiting time distribution

A time between two successive arrivals is called as waiting time and it is a random variable. The waiting time probability distribution function is an important attribute of any arrival or counting random process. The waiting time probability distribution function $\psi(\tau)$ represents the probability density of event that an arrival is occurred at the time moment $t_k = t_{k-1} + \tau$ after the previous one happened at the moment t_{k-1}. Therefore, the probability that interarrival time $t_k - t_{k-1}$ between the successive arrivals satisfies $\tau \leqslant t_k - t_{k-1} \leqslant \tau\,\mathrm{d}\tau$ is equal to $\psi(\tau)\,\mathrm{d}\tau$. Further, the probability $P(\tau)$ that a given interarrival time is greater or equal to τ can be expressed in term of probability distribution function $\psi(\tau)$ by the following way:

$$P(\tau) = \int_\tau^\infty \mathrm{d}t'\,\psi(t') = 1 - \int_0^\tau \mathrm{d}t'\,\psi(t'), \tag{15}$$

or

$$\psi(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}P(\tau). \tag{16}$$

It is easy to see that $\int_0^\tau \mathrm{d}t'\,\psi(t')$ is the probability of at least one arrival at any moment in the interval $[0, \tau]$. To evaluate this probability we apply Eq. (14). Then we write

$$\int_0^\tau \mathrm{d}t'\,\psi(t') = \sum_{n=1}^{\infty} P(n,\tau) = 1 - \mathrm{e}^{-\overline{n}\tau}. \tag{17}$$

By combining Eqs. (15)–(17) we finally find

$$\psi(\tau) = \overline{n}\mathrm{e}^{-\overline{n}\tau}. \tag{18}$$

The exponential distribution is manifestation of the Markov property of the Poisson random process.

3. The fractional Poisson process

3.1. Generation function

We introduce the fractional Poisson process as the counting process with probability $P_\mu(n,t)$ of arriving n items ($n = 0, 1, 2, \ldots$) by time t. The probability $P_\mu(n,t)$ is governed by the following special form of the fractional Kolmogorov–Feller equation:

$${}_0D_t^\mu P_\mu(n,t) = \nu\big(P_\mu(n-1,t) - P_\mu(n,t)\big) + \frac{t^{-\mu}}{\Gamma(1-\mu)}\delta_{n,0}, \quad 0 < \mu \leqslant 1, \tag{19}$$

with normalization condition

$$\sum_{n=0}^{\infty} P_\mu(n,t) = 1, \tag{20}$$

where the operator of fractional derivation ${}_0D_t^\mu$ is defined as the Riemann–Liouville fractional integral, [1]

$${}_0D_t^\mu f(t) = \frac{1}{\Gamma(-\mu)} \int_0^t \frac{\mathrm{d}\tau f(\tau)}{(t-\tau)^{1+\mu}},$$

and $\delta_{n,0}$ is the Kronecker symbol, the gamma function $\Gamma(\mu)$ has the familiar representation $\Gamma(\mu) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t} t^{\mu-1}$, $Re\mu > 0$, and parameter v has physical dimension $[v] = \sec^{-\mu}$. The initial condition $P_\mu(n, t=0) = \delta_{n,0}$ is incorporated in Eq. (19). One can consider the fractional differential–difference equation (19) as generalization of Eqs. (6) and (7).

To solve Eq. (19) it is convenient to use the method of generating function. Namely, we introduce the fractional generating function $G_\mu(s,t)$

$$G_\mu(s,t) = \sum_{n=0}^{\infty} s^n P_\mu(n,t). \tag{21}$$

Then by multiplying Eq. (19) by s^n, summing over n we obtain the following fractional differential equation for the generating function $G_\mu(s,t)$:

$$\begin{aligned} {}_0D_t^\mu G_\mu(s,t) &= v\left(\sum_{n=0}^{\infty} s^n P_\mu(n-1,t) - \sum_{n=0}^{\infty} s^n P_\mu(n,t)\right) + \frac{t^{-\mu}}{\Gamma(1-\mu)} \\ &= v(s-1)G_\mu(s,t) + \frac{t^{-\mu}}{\Gamma(1-\mu)}. \end{aligned} \tag{22}$$

The solution of this fractional equation has a form

$$G_\mu(s,t) = E_\mu(vt^\mu(s-1)), \tag{23}$$

where $E_\mu(z)$ is the Mittag-Leffler function given by its series representation [13–15]

$$E_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\mu m + 1)}. \tag{24}$$

The Mittag-Leffler function $E_\mu(z)$ can be considered as fractional generalization of the well known exponential function $\exp(z)$. It is easy to see that at $\mu = 1$ the Mittag-Leffler function $E_\mu(z)$ becomes the exponential function, $E_1(z) = \mathrm{e}^z$, and Eq. (23) is transformed into Eq. (13) for the generation function of the standard Poisson random process. The parameter v becomes $\overline{n}$ at $\mu = 1$.

Expanding (23) in series over s results in accordance with definition (21)

$$P_\mu(n,t) = \frac{(vt^\mu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-vt^\mu)^k}{\Gamma(\mu(k+n)+1))}, \quad 0 < \mu \leqslant 1. \tag{25}$$

[1] The basic formulas on fractional calculus can be found in Refs. [2,3,10–12].

The $P_\mu(n,t)$ gives us the probability that in the time interval $[0,t]$ we observe n events. When $\mu = 1$ the $P_\mu(n,t)$ is transformed to the standard Poisson distribution, see Eq. (14). Thus, Eq. (25) can be considered as fractional generalization of the standard Poisson distribution. The presence of additional parameter μ brings new features in comparison with the standard Poisson distribution. We investigate them in Sections 3.2–3.4.

Let us show that $P_\mu(n,t)$ satisfies the normalizing condition $\sum_{n=0}^{\infty} P_\mu(n,t) = 1$. Indeed, one can see that the chain of equations holds

$$\sum_{n=0}^{\infty} P_\mu(n,t) = \sum_{n=0}^{\infty} \frac{(\nu t^\mu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))} = \sum_{n=0}^{\infty} \frac{(\nu t^\mu)^n}{n!} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} \frac{(-\nu t^\mu)^{k-n}}{\Gamma(\mu k+1)}$$
$$= \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\mu k+1)} \sum_{n=0}^{k} \frac{(\nu t^\mu)^n (-\nu t^\mu)^{k-n}}{n!(k-n)!} = \sum_{k=0}^{\infty} \frac{(\nu t^\mu)^k (1-1)^k}{\Gamma(\mu k+1)} = \frac{1}{\Gamma(1)} = 1.$$

3.2. Mean and variance

The mean $\overline{n}_\mu$ of the fractional Poisson process can be calculated straightforwardly

$$\overline{n}_\mu = \sum_{n=0}^{\infty} n P_\mu(n,t) = \frac{\nu t^\mu}{\Gamma(\mu+1)}. \tag{26}$$

The second order moment $\overline{n^2}_\mu$ is given by

$$\overline{n^2_\mu} = \sum_{n=0}^{\infty} n^2 P_\mu(n,t) = \overline{n}_\mu + \overline{n}_\mu^2 \frac{\sqrt{\pi}\Gamma(\mu+1)}{2^{2\mu-1}\Gamma(\mu+\frac{1}{2})}. \tag{27}$$

Then variance of the fractional Poisson process is

$$\sigma_\mu = \overline{n^2_\mu} - \overline{n}^2_\mu = \overline{n}_\mu + \overline{n}^2_\mu \left\{ \frac{\mu B\left(\mu, \frac{1}{2}\right)}{2^{2\mu-1}} - 1 \right\}, \tag{28}$$

where $B(\mu, \frac{1}{2})$ is the Beta-function defined as [14]

$$B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

It is easy to see that at $\mu = 1$ Eq. (28) is transformed to

$$\sigma_1 = \overline{n}_1 = \overline{n}t, \tag{29}$$

here $\overline{n}$ is the rate of the standard Poisson process. The Eq. (29) express the well known property that variance of the standard Poisson process is equal to its mean [4].

Finally note, that the probability distribution of the fractional Poisson process can be represented in the terms of the Mittag-Leffler function $E_\mu(z)$ by the following compact way:

$$P_\mu(n,t) = \frac{(-z)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} E_\mu(z) \bigg|_{z=-\nu t^\mu} \tag{30}$$

$$P_\mu(n=0,t) = E_\mu(-\nu t^\mu). \tag{31}$$

At $\mu = 1$ Eqs. (30) and (31) are transformed to the well known equations for the standard Poisson process with the substitution $v \to \overline{n}$.

3.3. Moment generation function

The equation for the moment of any integer order of the fractional Poisson can be easily found by means of the moment generation function $H_\mu(s,t)$ which is defined as

$$H_\mu(s,t) = \sum_{n=0}^{\infty} \mathrm{e}^{-sn} P_\mu(n,t). \tag{32}$$

Indeed, for the moment of kth order we have

$$\overline{n_\mu^k} = (-1)^k \frac{\partial^k H_\mu(s,t)}{\partial s^k}\bigg|_{s=0}. \tag{33}$$

By multiplying Eq. (19) by e^{-sn}, summing over n as a result we find the following fractional differential equation for the moment generating function $H_\mu(s,t)$:

$$\begin{aligned} {}_0D_t^\mu H_\mu(s,t) &= v\left(\sum_{n=0}^{\infty} \mathrm{e}^{-sn} P_\mu(n-1,t) - \sum_{n=0}^{\infty} \mathrm{e}^{-sn} P_\mu(n,t)\right) + \frac{t^{-\mu}}{\Gamma(1-\mu)} \\ &= v(\mathrm{e}^{-s}-1)H_\mu(s,t) + \frac{t^{-\mu}}{\Gamma(1-\mu)}. \end{aligned} \tag{34}$$

The solution can be written as

$$H_\mu(s,t) = E_\mu(vt^\mu(\mathrm{e}^{-s}-1)), \tag{35}$$

or in a series form

$$H_\mu(s,t) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\mu+1)} (vt^\mu(\mathrm{e}^{-s}-1))^m,$$

where the definition (24) of the Mittag-Leffler function was taken into account. Let us calculate, for example, the first order moment. We write

$$\overline{n}_\mu = -\frac{\partial H_\mu(s,t)}{\partial s}\bigg|_{s=0} = \sum_{m=1}^{\infty} \frac{m(vt^\mu)^m}{\Gamma(m\mu+1)} (\mathrm{e}^{-s}-1)^{m-1} \mathrm{e}^{-s}\bigg|_{s=0} = \frac{vt^\mu}{\Gamma(\mu+1)},$$

where we use the fact that the term $m = 1$ only contributes to sum over m.

At $\mu = 1$ Eq. (35) is transformed into well known equation for the moment generation function of the standard Poisson. Indeed, if we note that $E_1(z) = \mathrm{e}^z$, then we find the well known expression

$$H_1(s,t) = E_1(\overline{n}t(\mathrm{e}^{-s}-1)) = \exp\{\overline{n}t(\mathrm{e}^{-s}-1)\} \equiv H(s,t).$$

3.4. Waiting time distribution for fractional Poisson process

We introduce waiting time probability distribution function $\psi_\mu(\tau)$ of the fractional Poisson process by the way

$$\psi_\mu(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}P_\mu(\tau), \tag{36}$$

where $P_\mu(\tau)$ is the probability that a given interarrival time is greater or equal to τ

$$P_\mu(\tau) = 1 - \sum_{n=1}^{\infty} P_\mu(n,\tau) = E_\mu(-\nu\tau^\mu), \tag{37}$$

and $P_\mu(n,\tau)$ is given by Eq. (25). From Eqs. (36) and (37) we obtain

$$\psi_\mu(\tau) = \nu\tau^{\mu-1}E_{\mu,\mu}(-\nu\tau^\mu), \quad t \geqslant 0, \quad 0 < \mu \leqslant 1, \tag{38}$$

where the generalized two-parameter Mittag-Leffler function is [14]

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad E_{\alpha,1}(z) = E_\alpha(z). \tag{39}$$

The $\psi_\mu(\tau)$ defined by Eq. (38) is fractional generalization of the well known exponential probability distribution function $\psi(\tau)$ given by Eq. (18) to which it reduces for $\mu = 1$ because of $E_{1,1}(z) = \mathrm{e}^z$.

It occurs to be possible to obtain an explicit form of the fractional waiting time probability distribution function $\psi_{1/2}(\tau)$ at $\mu = 1/2$. In fact in accordance with definition (39) we have

$$E_{\frac{1}{2},\frac{1}{2}}(-z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} + \sum_{k=0}^{\infty} \frac{(-z)^{k+1}}{\Gamma\left(\frac{k}{2}+1\right)} = \frac{1}{\sqrt{\pi}} - zE_{\frac{1}{2}}(-z). \tag{40}$$

Further, the Mittag-Leffler function $E_{\frac{1}{2}}(-z)$ has the representation [14]

$$E_{\frac{1}{2}}(-z) = \mathrm{e}^{z^2}\mathrm{erf}c(z), \tag{41}$$

where

$$\mathrm{erf}c(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{d}y\, \mathrm{e}^{-y^2} \tag{42}$$

is the complementary error function.

Substituting Eqs. (41) and (42) into Eq. (40) and then putting Eq. (40) with $z = \nu\tau^\mu$ into (38) give the representation for fractional waiting time probability distribution function $\psi_{1/2}(\tau)$

$$\psi_{1/2}(\tau) = \frac{\nu}{\sqrt{\pi\tau}} - \frac{2\nu^2\mathrm{e}^{\nu^2\tau}}{\sqrt{\pi}} \int_{\nu\sqrt{\tau}}^\infty \mathrm{d}y\, \mathrm{e}^{-y^2}. \tag{43}$$

The probability distribution function $\psi_\mu(\tau)$ has the following asymptotic behavior

$$\psi_\mu(\tau) \simeq \left| \begin{array}{ll} 1/\nu\tau^{\mu+1}, & \tau \to \infty, \\ \nu\tau^{\mu-1}, & \tau \to 0. \end{array} \right. \tag{44}$$

The power law asymptotic behavior is observed in probability distribution of duration of the Internet sessions at $\tau \to \infty$ [5].

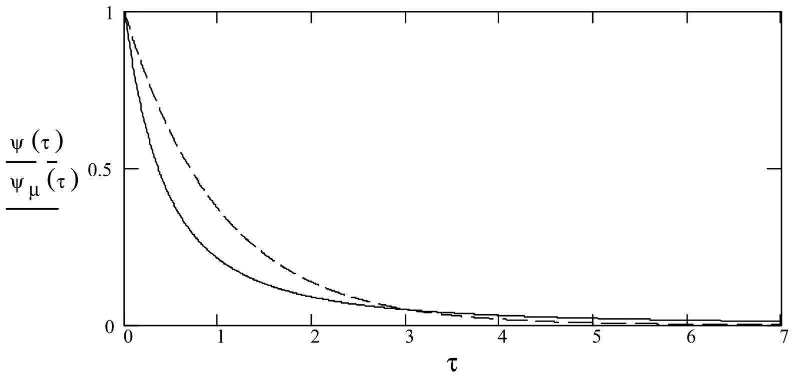

Fig. 1. The probability distribution functions $\psi(\tau)$ and $\psi_\mu(\tau)$ defined by Eqs. (18) and (38) reciprocally and evaluated at $\overline{n}=1$, $\nu=1$, $\mu=0.7$.

At last if we note that there exists the formula

$$\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-ut}t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)=\frac{1}{1+u^\alpha}, \tag{45}$$

then we easy get the Laplace transform $\psi_\mu(u)$ of the fractional waiting time distribution function (38)

$$\psi_\mu(u)=\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-u\tau}\psi_\mu(\tau)=\nu\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-u\tau}\tau^{\mu-1}E_{\mu,\mu}(-\nu\tau^\mu)=\frac{\nu}{\nu+u^\mu},\quad 0<\mu\leqslant 1 \tag{46}$$

which was originally proposed for $\nu=1$ in [3], see Eq. (7.5) as fractional generalization of the Laplace transform of the exponential distribution $\psi(u)=\nu/(\nu+u)$. Applying the inverse Laplace transform

$$\psi_\mu(\tau)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{d}u\,\mathrm{e}^{u\tau}\psi_\mu(u),$$

yields at $\nu=1$ Eq. (3) which coincides with Eq. (7.6) in [3].

Fig. 1 shows the plot of the probability distribution functions $\psi(\tau)$ and $\psi_\mu(\tau)$.

4. Fractional compound Poisson process

We call stochastic process $\{X(t),\ t\geqslant 0\}$ a fractional compound Poisson process if it is represented by

$$X(t)=\sum_{i=1}^{N(t)}Y_i, \tag{47}$$

where $\{N(t),\ t\geqslant 0\}$ is a fractional Poisson process, and $\{Y_i,\ i=1,2,\ldots\}$ is a family of independent and identically distributed random variables with probability distribution function $p(Y)$ for each Y_i. The process $\{N(t),\ t\geqslant 0\}$ and the sequence $\{Y_i,\ i=1,2,\ldots\}$ are assumed to be independent.

We now calculate the moment generation function $J_\mu(s,t)$ of fractional compound Poisson process

$$J_\mu(s,t) = \langle \exp\{sX(t)\}\rangle_{Y_i,N(t)}, \tag{48}$$

where $\langle\ldots\rangle_{Y_i,N(t)}$ is averaging procedure which includes two statistically independent averaging procedures:

1. Averaging over independent random variables Y_i, $\langle\ldots\rangle_{Y_i}$

$$\langle\ldots\rangle_{Y_i} = \int \mathrm{d}Y_1\ldots\mathrm{d}Y_n p(Y_1)\ldots p(Y_n)\ldots, \tag{49}$$

   where $p(Y_i)$ is the probability density of random variable Y_i.
2. Averaging over random number n governed by the fractional Poisson process

$$\langle\ldots\rangle_{N(t)} = \sum_{n=0}^{\infty} P_\mu(n,t)\ldots, \tag{50}$$

   where $P_\mu(n,t)$ is given by Eq. (25).

One can see from Eq. (48) that kth order moment is obtained by differentiating $J_\mu(s,t)$ k times with respect to s, then putting $s=0$, that is

$$\langle X^k(t)\rangle_{Y_i,N(t)} = \frac{\partial^k}{\partial s^k} J_\mu(s,t)\bigg|_{s=0}. \tag{51}$$

To obtain equation for the moment generation function $J_\mu(s,t)$ we apply Eqs. (49) and (50) to Eq. (48)

$$\begin{aligned}
J_\mu(s,t) &= \sum_{n=0}^{\infty}\langle \exp\{sX(t)|N(t)=n\}\rangle_{Y_i} P_\mu(n,t) \\
&= \sum_{n=0}^{\infty}\langle \exp\{s(Y_1+\ldots+Y_n)\}\rangle_{Y_i} \times \frac{(\nu t^\mu)^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))} \\
&= \sum_{n=0}^{\infty}\langle \exp\{s(Y_1)\}\rangle_{Y_i}^n \times \frac{(\nu t^\mu)^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))},
\end{aligned} \tag{52}$$

where we used the independence of $\{Y_1, Y_2, \ldots\}$ and $N(t)$ and the independence of the Y_i's between themselves. Hence, letting

$$g(s) = \langle \mathrm{e}^{sY}\rangle_Y \tag{53}$$

for the moment generation function of random variables Y_i, we find from Eq. (52) the moment generation function $J_\mu(s,t)$ of the fractional compound Poisson process

$$J_\mu(s,t) = \sum_{n=0}^{\infty} g^n(s) \times \frac{(\nu t^\mu)^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))} = E_\mu(\nu t^\mu(g(s)-1)). \tag{54}$$

Upon differentiation of the above, it easy follows, for example, that the mean of the fractional compound Poisson process is

$$\langle X(t)\rangle_{Y_i,N(t)} = \frac{\partial}{\partial s} J_\mu(s,t)\bigg|_{s=0} = \langle Y\rangle_Y \frac{vt^\mu}{\Gamma(\mu+1)}, \tag{55}$$

which is manifestation of independency of fractional Poisson process and random variables Y_i.

5. Conclusions

To explain empirically observed power law asymptotic behavior of probability distribution function of interarrival times we propose non-Markov fractional Poisson model based on fractional generalization of the Kolmogorov–Feller equation. We obtain analytical expression for the probability $P_\mu(n,t)$ (see Eq. (25)) that in the time interval $[0,t]$ we observe n events governed by fractional Poisson stream with fractality parameter μ, $0<\mu\leqslant 1$. The probability $P_\mu(n,t)$ is fractional generalization of the well known Poisson probability and is transformed to it at $\mu=1$. Because of property of the Mittag-Leffler function $E_1(z)=\mathrm{e}^z$ the Markov property of the standard Poisson process is reconstructed at $\mu=1$. Thus, all new general equations developed in the paper in the limit case $\mu=1$ are transformed into the well known equations concerned the standard Poisson random process.

One of an important feature of a counting random process is statistics of its interarrival times. It is well known that the Poisson model predicts the exponential probability distribution of interarrival times. The developed fractional Poisson model predicts power law behavior of interarrival times probability distribution (see Eqs. (38) and (44)). The network communication traffic is an example of counting process with power law asymptotics of interarrival time distribution. Thus, the fractional Poisson process can be applied to statistical analysis data collected from different network communication systems.

To show how one can work with the fractional Poisson process we define the fractional compound Poisson process and obtain analytical expression for its moment generation function.

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