ON SOLVABILITY OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES*

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Abstract

We describe a set of initial data in the abstract Cauchy problem for the linear equation with the Caputo fractional derivative and an unbounded linear closed operator A in a Banach space X

$$\begin{array}{ll} D_*^\alpha x(t) = Ax(t), & m-1 < \alpha \le m \in \mathbf{N}, \\ \frac{d^k}{dt^k} x(t)|_{t=0} = \xi_k, & k = 0, \dots, m-1 \end{array}$$

for which the corresponding solutions can be represented by means of the Mittag-Leffler operator function. Some properties of the Mittag-Leffler operator function are given.

Mathematics Subject Classification: primary 34G10, secondary 26A33, 45J05, 45K05

Key Words and Phrases: Abstract Cauchy problem, Caputo fractional derivative, Mittag-Leffler function, Roumieu space, Gevrey space, Beurling space

1. Introduction

Ordinary and partial differential equations of fractional order (with the fractional derivatives in Caputo, Riemann-Liouville or inverse Riesz potential sense) have excited in recent years a considerable interest both in mathematics and in applications (see [4], [6]-[14], [17], [19] and references there). In mathematical

* Partially supported by the Research Commission of Free University of Berlin (Project "Convolutions")

treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order α ($\alpha \geq 0$) is normally used:

$$D^{\alpha}x(t) := \left(\frac{d}{dt}\right)^{m} J^{m-\alpha}x(t),\ t > 0,\ m-1 < \alpha \leq m \in \mathbf{N}, \tag{1}$$

where

$$J^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau,\ \alpha > 0,\ t > 0, \tag{2}$$
$$J^0 x(t) := x(t),\ t > 0$$

is the Riemann-Liouville fractional integral of order α. The Riemann-Liouville fractional derivative is left-inverse (and not right-inverse) to the corresponding fractional integral, which is the natural generalization of the Cauchy formula for the n-fold primitive of a function $x(t)$. In formulation of the initial value problems for ordinary (or the Cauchy problem for partial) differential equations of fractional order α with the fractional derivatives in the Riemann-Liouville sense (see, for example, [10], [13], [17], [19]) the initial conditions are given in terms of the fractional integrals $J^{m-k-\alpha}x^{(k)}(0+)$, $k = 0, 1, \ldots, m-1$. On the other hand, in modeling of real processes the initial conditions are normally expressed in terms of a given number of bounded initial values assumed by the field variable x and its derivatives of integer order. In order to meet this physical requirement, an alternative definition of fractional derivative was introduced by Caputo [3] and adopted by Caputo and Mainardi in the framework of the theory of linear viscoelasticity:

$$D_*^{\alpha}x(t) := J^{m-\alpha}x^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \tag{3}$$
$$m-1 < \alpha \leq m \in \mathbf{N},\ t > 0.$$

In the papers [2], [9] the Caputo derivative (3) in the case $0 < \alpha < 1$ was called the regularized fractional derivative of order α.

Using ideas related to the theory of first and second order abstract differential equations, some results have been obtained for the abstract Cauchy problem for the linear fractional differential equation

$$\begin{array}{ll} D_*^{\alpha}x(t) = Ax(t), & m-1 < \alpha \leq m \in \mathbf{N}, \quad 0 < t < T \leq +\infty \\ \frac{d^k}{dt^k}x(t)|_{t=0} = \xi_k, & k = 0, \ldots, m-1, \end{array} \tag{4}$$

where $A : \mathcal{D}(A) \to X$, $\mathcal{D}(A) \subset X$ is a linear unbounded closed operator in a Banach space X. In the papers [2], [9] the necessary and sufficient conditions for solvability of the Cauchy problem (4) in the case $0 < \alpha < 1$ were given, extending the conditions of the Hille-Yosida theorem from $\alpha = 1$ to $\alpha \in (0, 1]$. The case

$0 < \alpha \le 2$ was considered in [4]. In the paper [10] the solvability conditions for the abstract Cauchy problem (4) with the Riemann-Liouville fractional derivative instead of the Caputo derivative and the initial conditions given in terms of the fractional integrals $J^{m-k-\alpha}x^{(k)}(0+),\ k = 0, 1, \dots, m-1$ have been obtained in the case $\alpha > 0$.

The other, equivalent approach to the Cauchy problem (4) consists in the reduction of this problem to an evolutionary integral equation of the type

$$x(t) = f(t) + \int_0^t k(t-\tau)Ax(\tau)\,d\tau, \quad t \ge 0, \tag{5}$$

where $k \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ is a scalar kernel. The general theory of such equations (also in the non-scalar case, i.e., $k(s) \equiv 1,\ s \in \mathbf{R}_+$ and an operator $A(t-\tau)$ depends on $t-\tau$) was presented in [18]. The special case of the kernel $k(s) = s^{\alpha-1}/\Gamma(\alpha)$ and the differential operator A of first order for $0 < \alpha \le 1$, of second order for $1 < \alpha \le 2$, in a Hilbert space was considered in details in [4], [20]. The last reference contains also investigations of numerical methods for this problem.

It is known ([12]) that if $X = \mathbf{R}$ or $X = \mathbf{C}$ and the operator A can be identified with a multiplicative constant the unique solution of the Cauchy problem (4) for $t \ge 0$ is given by the formula

$$x(t) = \sum_{k=0}^{m-1} \xi_k x_k(t),\ \ x_k(t) = t^k E_{\alpha,k+1}(At^\alpha) \tag{6}$$

with the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)},\ \ z \in \mathbf{Z}. \tag{7}$$

The partial solutions $x_k(t),\ k = 0, 1, \dots, m-1$ can be also represented in the form ([6], [12])

$$x_k(t) = J^k u_0(t),\ \ u_0(t) = E_{\alpha,1}(At^\alpha) := E_\alpha(At^\alpha),$$

where J^k is the k-fold integral (replace α by k in (2)) and

$$E_\alpha(z) := E_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} \tag{8}$$

is the classical Mittag-Leffler function.

Let now X be a general Banach space and A be a linear unbounded closed operator on this space. Using the methods, given in [1], [16] for the case $\alpha = 1$ we describe those initial data of the Cauchy problem (4) for which the solutions may be also represented in the form (6). This description is done in terms of so-called Roumieu spaces generated by the operator A and their inductive and projective limits, the Gevrey and Beurling spaces.

2. Linear equations of fractional order

In the case of an unbounded operator A the right-hand side of the relation (6) is not defined on the whole Banach space X. We can however investigate the conditions for $\xi = (\xi_0, \ldots, \xi_{m-1}),\ \xi_k \in X,\ k = 0, \ldots, m-1$ under which the time-dependent operator $\mathcal{E}_\alpha(At^\alpha) : X^m \to X$ with

$$\mathcal{E}_\alpha(At^\alpha)\xi := \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)\xi_k, \tag{9}$$

$$t^k E_{\alpha,k+1}(At^\alpha)\xi_k := \sum_{n=0}^{\infty} \frac{t^{\alpha n+k} A^n \xi_k}{\Gamma(1+k+\alpha n)}, \quad m-1 < \alpha \le m \in \mathbf{N}$$

is well defined. We shall refer to the operator $\mathcal{E}_\alpha(At^\alpha)$ given by (9) as to the Mittag-Leffler operator function.

We prove that for such $\xi = (\xi_0, \ldots, \xi_{m-1})$ the formula (9) defines a solution of the Cauchy problem (4). The description of the set of the initial conditions for which the Cauchy problem (4) has a solution in the form (9) can be given in terms of the Roumieu spaces generated by the operator A (see [1], [5], [16]).

Definition 2.1. Let A be a linear unbounded operator in a Banach space X, $\mu = \{\mu_n\}_0^\infty$ be a sequence with $\mu_0 = 1,\ \mu_n > 0,\ n = 1, 2 \ldots$, and $0 < L < \infty$. The Roumieu space $\mathcal{R}(A, \mu, L)$ is defined as the set of elements $x \in X$ such that

$$\sup_{0 \le n < \infty} \frac{\|A^n x\|}{L^n \mu_n} < \infty.$$

Equipped with the norm

$$\|x\|_{\mathcal{R}(A,\mu,L)} = \sup_{0 \le n < \infty} \frac{\|A^n x\|}{L^n \mu_n}, \tag{10}$$

this space is a Banach space continuously embedded in the space X.

We shall also use the Beurling and Gevrey spaces

$$\mathcal{B}(A, \mu) = \bigcap_{0<L<\infty} \mathcal{R}(A, \mu, L), \quad \mathcal{G}(A, \mu) = \bigcup_{0<L<\infty} \mathcal{R}(A, \mu, L).$$

These spaces equipped with the topologies of inductive and projective limits, respectively, are locally convex spaces.

One of the most important problems in the theory of the Roumieu, Beurling and Gevrey spaces in connection with the abstract Cauchy problem for differential equations of fractional order is the problem of density of these spaces in the original space X. For the survey of some results for the classical case $\mu = \{(n!)^\alpha\}_0^\infty,\ 0 \le$

$\alpha < \infty$ see, e.g., the paper [1]. It should be noted that most results in this direction have been obtained for the cases $\alpha = 0$, $\alpha = 1$, and $\alpha > 1$. Few if any could be found in the case $0 < \alpha < 1$, and this case should be yet investigated.

In this article we shall deal with the Roumieu spaces with the sequence $\mu = \{\Gamma(1+\alpha n)\}_0^\infty$, $\alpha \geq 0$, which will be denoted as $\mathcal{R}_\alpha(A, L)$. The connection of these spaces to the classical Roumieu spaces $\tilde{\mathcal{R}}_\alpha(A, L) := \mathcal{R}(A, \mu, L)$ with $\mu = \{(n!)^\alpha\}_0^\infty$ is given by the following simple results.

LEMMA 2.1. *The inclusions*

$$\tilde{\mathcal{R}}_\alpha(A, L_2) \subseteq \mathcal{R}_\alpha(A, L) \subseteq \tilde{\mathcal{R}}_\alpha(A, L_1)$$

hold true for $\alpha \geq 0$ *and every* $L,\ L_1$, *and* L_2 *if* $L_1 > \alpha^\alpha L > L_2$.

P r o o f. The case $\alpha = 0$ is evident. For $\alpha > 0$, making use of Stirling's asymptotic formula for the gamma function (see, for example, [15]), we get the estimate

$$\frac{(n!)^\alpha}{\Gamma(1+\alpha n)} \leq C_1 n^{\alpha/2-1/2} \alpha^{-\alpha n},\ n \geq 1,$$

with a constant C_1 depending on α. This estimate implies the inclusions (11) and (12):

$$\tilde{\mathcal{R}}_\alpha(A, L_2) \subseteq \mathcal{R}_\alpha(A, L) \text{ for every } L_2 < \alpha^\alpha L, \tag{11}$$

$$\mathcal{R}_\alpha(A, L) \subseteq \tilde{\mathcal{R}}_\alpha(A, L_1) \text{ for every } L_1 > \alpha^\alpha L. \tag{12}$$

Combining (11) and (12) we get Lemma 2.1. ∎

As a direct consequence of the previous Lemma and the definitions of the Beurling and Gevrey spaces we get the following result:

LEMMA 2.2. *For the Beurling and Gevrey spaces* $\mathcal{B}_\alpha(A)$, $\mathcal{G}_\alpha(A)$, $\tilde{\mathcal{B}}_\alpha(A)$, $\tilde{\mathcal{G}}_\alpha(A)$ *generated by the families of the Roumieu spaces* $\mathcal{R}_\alpha(A, L)$ *and* $\tilde{\mathcal{R}}_\alpha(A, L)$, *respectively, we have the identities*

$$\mathcal{B}_\alpha(A) = \tilde{\mathcal{B}}_\alpha(A),\ \mathcal{G}_\alpha(A) = \tilde{\mathcal{G}}_\alpha(A). \tag{13}$$

We describe now a set of initial data of the Cauchy problem (4) in a Banach space X for which the solutions can be represented in the form (9). We do this in terms of the Roumieu spaces $\mathcal{R}_\alpha(A, L)$ generated by the operator A.

THEOREM 2.1. *Let* A *be a linear unbounded closed operator in a Banach space* X *and* $\xi_0, \ldots, \xi_{m-1} \in X$.

If a solution of the form (9) *of the Cauchy problem* (4) *exists on the interval* $[0, L]$ *then* $\xi_k \in \mathcal{R}_\alpha(A, L_1),\ k = 0, \ldots, m-1$ *for every* $L_1 > L^{-\alpha}$.

Conversely, if for some L, $\xi_k \in \mathcal{R}_\alpha(A, L),\ k = 0, \ldots, m-1$, *then the Mittag-Leffler operator function given by* (9) *defines a solution of the Cauchy problem* (4) *on the interval* $[0, L^{-1/\alpha})$.

P r o o f. Let the function

$$x(t)=\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{t^{\alpha n+k}A^n\xi_k}{\Gamma(1+k+\alpha n)}$$

with fixed $\xi_k\in X,\ k=0,\dots,m-1$ be defined on the interval $[0,L]$. Then

$$\lim_{n\to\infty}\frac{t^{\alpha n+k}A^n\xi_k}{\Gamma(1+k+\alpha n)}=0,\ 0\le t\le L,\ k=0,\dots,m-1,$$

and, consequently,

$$\lim_{n\to\infty}\frac{\|A^n\xi_k\|\,L^{\alpha n}}{\Gamma(1+k+\alpha n)}=0,\ k=0,\dots,m-1.$$

Using the asymptotic formula [15]

$$\frac{\Gamma(a+s)}{\Gamma(b+s)}=s^{a-b}[1+O(1/s)],\ |\arg s|\le\pi-\delta,\ 0<\delta<\pi,\ |s|\to\infty, \tag{14}$$

we arrive for every $L_1>L^{-\alpha}$ at the relation

$$\lim_{n\to\infty}\frac{\|A^n\xi_k\|}{L_1^n\Gamma(1+\alpha n)}=\lim_{n\to\infty}\frac{\|A^n\xi_k\|\,L^{\alpha n}}{\Gamma(1+k+\alpha n)}$$

$$\times\lim_{n\to\infty}\left(\frac{1}{L_1L^\alpha}\right)^n\frac{\Gamma(1+k+\alpha n)}{\Gamma(1+\alpha n)}=0,\ k=0,\dots,m-1,$$

which implies the inclusion $\xi_k\in\mathcal{R}_\alpha(A,L_1),\ k=0,\dots,m-1$.

Conversely, let $\xi_k\in\mathcal{R}_\alpha(A,L),\ k=0,\dots,m-1$. Since $\Gamma(1+\alpha n)\le\Gamma(1+k+\alpha n),\ k=0,\dots,m-1,\ n\in\mathbf{N}$, the series

$$\sum_{n=0}^{\infty}\frac{t^{\alpha n+k}A^n\xi_k}{\Gamma(1+k+\alpha n)},\ k=0,\dots,m-1$$

is absolutely and uniformly convergent on every compact subset of the interval $[0,L^{-1/\alpha})$. Hence this series defines for every fixed $t\in[0,L^{-1/\alpha})$ a linear operator (we denote it by $t^kE_{\alpha,k+1}(At^\alpha)$) from $\mathcal{R}_\alpha(A,L)$ into X. If we fix an element $\xi_k\in\mathcal{R}_\alpha(A,L)$ then the function

$$t\mapsto x_k(t)=\sum_{n=0}^{\infty}\frac{t^{\alpha n+k}A^n\xi_k}{\Gamma(1+k+\alpha n)},\ k=0,\dots,m-1 \tag{15}$$

is continuous on the interval $[0,L^{-1/\alpha})$. We also have

$$x_k(0)=\begin{cases}\xi_0, & k=0,\\ 0, & k=1,\dots,m-1.\end{cases}$$

Let us prove that $x_k \in C^{m-1}[0, L^{-1/\alpha}) \cap C^m(0, L^{-1/\alpha})$. Indeed, by formal differentiation of (15) we get the series

$$Q_0(t) = \sum_{n=1}^{\infty} \frac{(\alpha n) t^{\alpha n - 1} A^n \xi_0}{\Gamma(1 + \alpha n)} \quad \text{for} \quad k = 0,$$

$$Q_k(t) = \sum_{n=0}^{\infty} \frac{(\alpha n + k) t^{\alpha n + k - 1} A^n \xi_k}{\Gamma(1 + k + \alpha n)}, \ k = 1, \dots, m-1.$$

which are absolutely and uniformly convergent on every compact subset of the interval $(0, L^{-1/\alpha})$ ($[0, L^{-1/\alpha})$ in the case $m > 1$). Hence $\frac{d}{dt} x_k(t) = Q_k(t),\ k = 0, \dots, m-1$, and $x_k \in C^1(0, L^{-1/\alpha})$ ($x_k \in C^1[0, L^{-1/\alpha})$ in the case $m > 1$) for all $k = 0, \dots, m-1$. In the case $m > 1$ we can directly check that

$$\frac{d}{dt} x_k(t)|_{t=0} = \begin{cases} 0, & k = 0, \\ \xi_1, & k = 1, \\ 0, & k = 2, \dots, m-1. \end{cases}$$

Repeating the same arguments m times and using the formula

$$\frac{\Gamma(a+n)}{\Gamma(a)} = a \cdot \ldots \cdot (a + n - 1), \ n \in \mathbf{N}$$

we arrive at the expressions

$$\frac{d^m}{dt^m} x_k(t) = \sum_{n=1}^{\infty} \frac{(\alpha n + k) \cdot \ldots \cdot (\alpha n + k - m + 1) t^{\alpha n + k - m} A^n \xi_k}{\Gamma(1 + k + \alpha n)} \tag{16}$$

$$= \sum_{n=0}^{\infty} \frac{t^{\alpha n + \alpha + k - m} A^{n+1} \xi_k}{\Gamma(1 + k - m + \alpha + \alpha n)}, \ k = 0, \dots, m-1,$$

$$\frac{d^l}{dt^l} x_k(t)|_{t=0} = \delta_{lk} \xi_k, \ l, k = 0, \dots, m-1,$$

and the inclusion $x_k \in C^{m-1}[0, L^{-1/\alpha}) \cap C^m(0, L^{-1/\alpha}),\ k = 0, \dots, m-1$. Since $m - 1 < \alpha \leq m$, all the functions $t \mapsto \frac{d^m}{dt^m} x_k(t),\ k = 0, \dots, m-1$ are integrable on the interval $(0, L^{-1/\alpha})$.

Using the formula

$$(J^\mu p_\nu)(t) = \frac{\Gamma(1+\nu)}{\Gamma(1 + \nu + \mu)} t^{\nu + \mu}, \ \mu \geq 0, \ p_\nu(t) := t^\nu, \ \nu > -1,$$

and the expressions (16) we get $(m - 1 < \alpha \leq m)$

$$D_*^\alpha x_k(t) = J^{m-\alpha} \frac{d^m}{dt^m} x_k(t) = \sum_{n=0}^{\infty} \frac{\Gamma(1 + k - m + \alpha + \alpha n)}{\Gamma(1 + k + \alpha n)}$$

$$\times \frac{t^{\alpha n+k} A^{n+1} \xi_k}{\Gamma(1+k-m+\alpha+\alpha n)} = \sum_{n=0}^{\infty} \frac{t^{\alpha n+k} A^{n+1} \xi_k}{\Gamma(1+k+\alpha n)}, \ k = 1, \ldots, m-1.$$

On the other hand, since the operator A is closed, we get

$$Ax_k(t) = A \sum_{n=0}^{\infty} \frac{t^{\alpha n+k} A^n \xi_k}{\Gamma(1+k+\alpha n)} = \sum_{n=0}^{\infty} \frac{t^{\alpha n+k} A^{n+1} \xi_k}{\Gamma(1+k+\alpha n)}, \ k = 0, \ldots, m-1.$$

Summarizing the obtained results we see that the operator-function (9) defines a solution of the Cauchy problem (4) on the interval $[0, L^{-1/\alpha})$. ∎

COROLLARY 2.1. *The statement of Theorem* 2.1 *can be rewritten in terms of Gevrey and Beurling spaces. Namely, the Mittag-Leffler operator function* (9) *defines a solution of the Cauchy problem* (4) *on a suitable interval* $[0, L)$ *(on every interval* $[0, L)$*) if and only if* $\xi_k \in \mathcal{G}_\alpha(A), \ k = 0, \ldots, m-1$ *(*$\xi_k \in \mathcal{B}_\alpha(A), \ k = 0, \ldots, m-1$*).*

3. The Mittag-Leffler operator function

We consider now some properties of the Mittag-Leffler operator function $\mathcal{E}_\alpha(At^\alpha)$ given by (9). For the sake of simplicity we restrict ourselves to the case $0 < \alpha \leq 1$. In this case (identifying ξ with ξ_0) (9) goes over in

$$\mathcal{E}_\alpha(At^\alpha)\xi = E_\alpha(At^\alpha)\xi := \sum_{n=0}^{\infty} \frac{t^{\alpha n} A^n \xi}{\Gamma(1+\alpha n)}. \tag{17}$$

THEOREM 3.1. *Let A be a linear unbounded closed operator in a Banach space X, $0 < \alpha \leq 1$, $0 < L_1 < L_2$, $h(L_1, L_2) = L_1^{-1/\alpha}[1 - (L_1/L_2)^{1/\alpha}]^{1/\alpha}$.*

Then the right-hand side of the relation (17) *defines, for $t \in [0, h(L_1, L_2))$, a continuous linear operator $E_\alpha(At^\alpha)$ from $\mathcal{R}_\alpha(A, L_1)$ into $\mathcal{R}_\alpha(A, L_2)$ with the norm estimate*

$$\|E_\alpha(At^\alpha)\|_{\mathcal{L}(\mathcal{R}_\alpha(A,L_1), \mathcal{R}_\alpha(A,L_2))} \leq \left(\frac{L_2}{L_1}\right)^{\frac{1}{\alpha}}. \tag{18}$$

P r o o f. For $0 < L_1 < L_2$ the inequality $L_1^{-1/\alpha} > h(L_1, L_2)$ holds true (the number $h(L_1, L_2)$ is given in Theorem 3.1). Then, according to the proof of Theorem 2.1, $E_\alpha(At^\alpha) \in \mathcal{L}(\mathcal{R}_\alpha(A, L_1), X)$ for $t \in [0, h(L_1, L_2))$. Furthermore, we shall prove that the operator $E_\alpha(At^\alpha)$ acts from the space $\mathcal{R}_\alpha(A, L_1)$ into the space $\mathcal{R}_\alpha(A, L_2)$ with the norm estimate (18). We use the auxiliary inequality

$$\frac{\Gamma(\beta_1 + \alpha x)}{\Gamma(\beta_2 + \alpha x)} \leq \frac{\Gamma(\beta_1 + x)}{\Gamma(\beta_2 + x)}, \text{ if } 0 < \alpha \leq 1, \ 0 < \beta_2 \leq \beta_1, \ x \geq 0, \tag{19}$$

which is a simple consequence of the Gauss summation theorem for the hypergeometric function ${}_2F_1(z)$ ([15]):

$$
{}_2F_1(\beta_1-\beta_2, x-\alpha x; \beta_1+x; 1) = \frac{\Gamma(\beta_1+x)}{\Gamma(\beta_2+x)}\frac{\Gamma(\beta_2+\alpha x)}{\Gamma(\beta_1+\alpha x)} =
$$

$$
= 1 + \sum_{k=1}^{\infty} \frac{(\beta_1-\beta_2)_k(x-\alpha x)_k}{(\beta_1+x)_k k!} \geq 1,
$$

where $(y)_k = y(y+1)\dots(y+k-1)$ stands for Pochhammer's symbol. Let us prove (18). Using the inequality (19), the binomial series

$$
(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(1+k)} z^k, \ |z|<1,
$$

the inequality ($\xi \in \mathcal{R}_\alpha(A,L)$)

$$
\|A^n\xi\| \leq \|\xi\|_{\mathcal{R}_\alpha(A,L)} L^n \Gamma(1+\alpha n), \ n = 0, 1, \dots, \tag{20}
$$

following from (10), and some elementary evaluations, we get

$$
\|E_\alpha(At^\alpha)\xi\|_{\mathcal{R}_\alpha(A,L_2)} \leq \sup_{0\leq n<\infty} \frac{1}{L_2^n \Gamma(1+\alpha n)} \sum_{k=0}^{\infty} \frac{t^{\alpha k}\|A^{n+k}\xi\|}{\Gamma(1+\alpha k)}
$$

$$
\leq \|\xi\|_{\mathcal{R}_\alpha(A,L_1)} \sup_{0\leq n<\infty} \left(\frac{L_1}{L_2}\right)^n \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha(n+k))}{\Gamma(1+\alpha n)\Gamma(1+\alpha k)} (h^\alpha(L_1,L_2)\cdot L_1)^k
$$

$$
\leq \|\xi\|_{\mathcal{R}_\alpha(A,L_1)} \sup_{0\leq n<\infty} \left(\frac{L_1}{L_2}\right)^n \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha n+k)}{\Gamma(1+\alpha n)\Gamma(1+k)} (h^\alpha(L_1,L_2)\cdot L_1)^k
$$

$$
= \|\xi\|_{\mathcal{R}_\alpha(A,L_1)} \sup_{0\leq n<\infty} \left(\frac{L_1}{L_2}\right)^n (1-h^\alpha(L_1,L_2)\cdot L_1)^{-1-\alpha n}
$$

$$
= \|\xi\|_{\mathcal{R}_\alpha(A,L_1)} (1-h^\alpha(L_1,L_2)\cdot L_1)^{-1} = \left(\frac{L_2}{L_1}\right)^{1/\alpha} \|\xi\|_{\mathcal{R}_\alpha(A,L_1)}.
$$

■

Remark 3.1. In the special case $\alpha = 1$ we have $h(L_1,L_2) = \frac{1}{L_1} - \frac{1}{L_2}$ and the norm estimate (18) in the form

$$
\|E_1(At)\|_{\mathcal{L}(\mathcal{R}_1(A,L_1),\mathcal{R}_1(A,L_2))} = \|e^{At}\|_{\mathcal{L}(\tilde{\mathcal{R}}_1(A,L_1),\tilde{\mathcal{R}}_1(A,L_2))} \leq \frac{L_2}{L_1},
$$

which is in accordance with the results of the paper [1]. If $\alpha \to 0$ then $h(L_1,L_2) \to +\infty$ for all fixed L_1, L_2 with $0 < L_1 < L_2$.

THEOREM 3.2. *Let A be a linear unbounded closed operator in a Banach space X, $0 < \beta < \alpha \leq 1$, and $L_1 < L_2$.*

Then the Mittag-Leffler operator function $E_\alpha(At^\alpha)$ given by (17) is a continuous linear operator from $\mathcal{R}_\beta(A, L_1)$ into $\mathcal{R}_\beta(A, L_2)$ for all $t \in [0, \infty)$, and

$$\|E_\alpha(At^\alpha)\|_{\mathcal{L}(\mathcal{R}_\beta(A,L_1),\mathcal{R}_\beta(A,L_2))} \leq \left(\frac{L_2}{L_1}\right)^{1/\beta} E_{\alpha-\beta}\left(\frac{t^\alpha L_1 L_2^{1/\beta}}{L_2^{1/\beta} - L_1^{1/\beta}}\right). \tag{21}$$

P r o o f. We need two auxiliary inequalities:

$$\frac{\Gamma(a+k)}{\Gamma(a)\Gamma(1+k)} \leq z^{-k}(1-z)^{-a} \text{ for all } a > 0,\ 0 < z < 1,\ k = 0, 1, \ldots, \tag{22}$$

and

$$\frac{\Gamma(1+b)}{\Gamma(1+a)} \leq \frac{1}{\Gamma(1+a-b)}, \text{ for all } a > b > 0. \tag{23}$$

To prove the first inequality we note that $(a > 0,\ 0 < z < 1)$

$$(1-z)^{-a} = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(a)\Gamma(1+j)} z^j \geq \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(1+k)} z^k,\ k = 0, 1, \ldots.$$

Using the Gauss summation theorem for the hypergeometric function ${}_2F_1(z)$ we get $(a > b > 0)$

$${}_2F_1(a-b, b; 1+a; 1) = \frac{\Gamma(1+a)\Gamma(1)}{\Gamma(1+b)\Gamma(1+a-b)}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(a-b)_k (b)_k}{(1+a)_k k!} \geq 1,$$

which proves (23). Now let $\xi \in \mathcal{R}_\beta(A, L_1)$ and $t \in [0, \infty)$. Using the inequalities (19), (20), (22) (with $z = 1 - (L_1/L_2)^{1/\beta}$), and (23), we have

$$\|E_\alpha(At^\alpha)\|_{\mathcal{R}_\beta(A,L_2)} \leq \sup_{0 \leq n < \infty} \frac{1}{L_2^n \Gamma(1+\beta n)} \sum_{k=0}^{\infty} \frac{\|A^{n+k}\xi\| t^{\alpha k}}{\Gamma(1+\alpha k)}$$

$$\leq \|\xi\|_{\mathcal{R}_\beta(A,L_1)} \sup_{0 \leq n < \infty} \left(\frac{L_1}{L_2}\right)^n \sum_{k=0}^{\infty} \frac{\Gamma(1+\beta(n+k))}{\Gamma(1+\beta n)\Gamma(1+\alpha k)} (t^\alpha L_1)^k$$

$$\leq \|\xi\|_{\mathcal{R}_\beta(A,L_1)} \sup_{0 \leq n < \infty} \left(\frac{L_1}{L_2}\right)^n \sum_{k=0}^{\infty} \frac{\Gamma(1+\beta n + k)\Gamma(1+\beta k)}{\Gamma(1+\beta n)\Gamma(1+k)\Gamma(1+\alpha k)} (t^\alpha L_1)^k$$

$$\leq \|\xi\|_{\mathcal{R}_\beta(A,L_1)} \sup_{0\leq n<\infty} \left(\frac{L_1}{L_2}\right)^n (1-z)^{-(1+\beta n)} \sum_{k=0}^{\infty} \frac{\Gamma(1+\beta k)}{\Gamma(1+\alpha k)} (t^\alpha L_1)^k z^{-k}$$

$$\leq \|\xi\|_{\mathcal{R}_\beta(A,L_1)} \frac{1}{1-z} \sum_{k=0}^{\infty} \frac{(t^\alpha L_1 z^{-1})^k}{\Gamma(1+(\alpha-\beta)k)}$$

$$= \|\xi\|_{\mathcal{R}_\beta(A,L_1)} \left(\frac{L_2}{L_1}\right)^{1/\beta} E_{\alpha-\beta}\left(\frac{t^\alpha L_1 L_2^{1/\beta}}{L_2^{1/\beta}-L_1^{1/\beta}}\right).$$

■

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