BOOLEAN ALGEBRA AND ITS APPLICATION

INCLUDING
BOOLEAN MATRIX ALGEBRA

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5. Plot the following functions on a Karnaugh map and hence determine whether they are or are not symmetric.

   (*a*) $f = (\bar{w} \cap x \cap \bar{y}) \cup (\bar{w} \cap \bar{y} \cap z) \cup (\bar{w} \cap x \cap y \cap z) \cup (w \cap x \cap \bar{y} \cap z)$
   (*b*) $f = (\bar{w} \cap x \cap y \cap z) \cup (\bar{w} \cap \bar{y}) \cup (\bar{x} \cap \bar{y} \cap \bar{z}) \cup (w \cap \bar{z})$
   (*c*) $f = (\bar{w} \cap x) \cup (y \cap z) \cup (\bar{w} \cap y) \cup (\bar{w} \cap \bar{x} \cap z) \cup (x \cap y \cap \bar{z}) \cup (w \cap x \cap \bar{y} \cap z)$

6. By the consideration of possible spheres in *n*-space, examine the symmetry of the functions given:

   (*a*) $f = 2 \cup 4 \cup 7 \cup 8 \cup 11 \cup 13$
   (*b*) $f = 0 \cup 9 \cup 10 \cup 12 \cup 23 \cup 24$
   (*c*) $f = 0 \cup 3 \cup 5 \cup 10 \cup 12 \cup 15 \cup 18 \cup 20 \cup 23 \cup 30 \cup 34 \cup 36 \cup 39 \cup 46 \cup 54 \cup 57$
   (*d*) $f = 3 \cup 5 \cup 6 \cup 8 \cup 9 \cup 10 \cup 15$

7. Obtain the switching functions in terms of joins of meets for each of the following:

   (*a*) $M(0)_2 \bigcup (v \cap w \cap \bar{x} \cap \bar{y} \cap \bar{z})$
   (*b*) $M(0)_3 \bigcup (v \cap w \cap x \cap y \cap z_1)$
   (*c*) $M(0)_{2,3} \bigcup (\bar{w} \cap x \cap \bar{y} \cap \bar{z})$

9

Matrices with Boolean Components—I

9.1 Introductory

The geometrical representation of switching functions in *n*-space rapidly loses its value as a suitable vehicle for minimization processes when the number of variables involved exceeds four. Even in the case of the four-cube, much of the simplicity of the method has disappeared. Much the same can be said of Karnaugh maps. Tabulation processes are also subject, though to a lesser degree, to a considerable increase in complexity as the number of variables becomes large. Some of these methods are to be preferred to others, but even the best can become excessively tedious for the minimization of functions of more than a dozen variables or so. It was therefore natural that investigations should be made into the possibility of extending the concepts of Boolean algebra to include matrices which have Boolean components. It is this extension of Boolean algebra which will form the basis of the considerations of this and a subsequent chapter. Basic definitions and the trivial case of the Karnaugh map as a matrix will be considered first and will then be followed by a survey of the algebra of matrices representing switching circuits. Applications of matrices to problems both of analysis and of synthesis will also be discussed.

9.2 Basic Considerations

A matrix is defined as a set of pq components a_{rs} arrayed in p rows and q columns which is subject to certain rules of combination with other similar such sets. A Boolean matrix may, in the first instance, be defined as such an array the components of which are all elements of a Boolean algebra, B. It is possible to arrive at such matrices in more than one way.

First, the entries upon a Karnaugh map may be considered as forming a Boolean matrix within the compass of the definition just given.

In this case, the components a_{rs} will each take either the value 1 or the value 0 accordingly as the minimal polynomials represented are present or absent in the function concerned. Every switching function will be represented by 2^n components in a square or a rectangular array. The meet of m such matrices, representing the series connection of m switching circuits, will be represented by a further matrix, the components of which are obtained by taking the meet of all corresponding components within the m matrices. Thus, if $A = [a_{rs}]$, $B = [b_{rs}]$, ... are the matrices representing the m switching functions which are in series connection, then the overall transmission function is given by

$$F = A \cap B \cap \ldots \cap M$$

where $$F = [f_{rs}]$$

and $$f_{rs} = a_{rs} \cap b_{rs} \cap \ldots \cap m_{rs}$$

In performing such an operation, it is clearly necessary that each of the m matrices should be of the same form. Thus, in the case of two elements the matrices must be either 4×1 or 2×2 in each case, and for four elements they must be either 16×1 or 8×2 or 4×4.

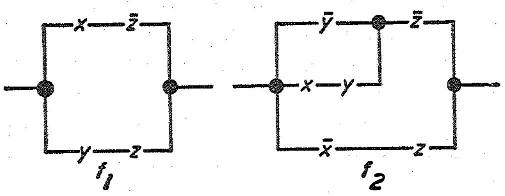

FIG. 9.2.1

As an example, consider the two switching circuits shown in figure 9.2.1. The corresponding Boolean functions are:

$$f_1(x, y, z) = (x \cap \bar{z}) \cup (y \cap z)$$
$$f_2(x, y, z) = (x \cap y \cap \bar{z}) \cup (\bar{x} \cap z) \cup (\bar{y} \cap \bar{z})$$

When these two circuits are placed in series, the matrix of the overall transmission function

$$f = f_1 \cap f_2$$

is given by

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

if the Karnaugh map is as shown in figure 9.2.2. The elements in the matrix of f are obtained by the application of

$$0 \cap 0 = 0$$
$$0 \cap 1 = 1 \cap 0 = 0$$
and $$1 \cap 1 = 1$$

to corresponding terms in the matrices of f_1 and f_2.

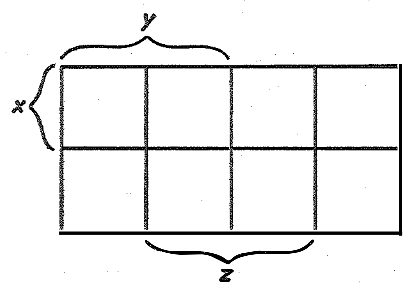

FIG. 9.2.2

In a similar manner, the parallel connection of m switching circuits yields an overall transmission matrix,

$$F = A \cup B \cup \ldots \cup M$$

where $$F = [f_{rs}]$$

and $$f_{rs} = a_{rs} \cup b_{rs} \cup \ldots \cup m_{rs}$$

The matrix for the transmission of the parallel connection of the circuits of figure 9.2.1, for example, is given by

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Here the elements of the matrix of $f = f_1 \cup f_2$ are obtained by the application of

$$0 \cup 0 = 0$$
$$0 \cup 1 = 1 \cup 0 = 1$$
and $$1 \cup 1 = 1$$

to corresponding terms in the matrices of f_1 and f_2.

It is possible to perform the cap and cup operations with matrices representing circuits having different numbers of variables. In such cases the functions with fewer variables have to be expanded so that n

is the same for all functions considered. As an example, consider the two functions represented by

$$f_1(y, z) = (y \cap z) \cup \bar{y}$$
$$f_2(w, x, y, z) = (y \cap z) \cup (\bar{x} \cap \bar{y} \cap z) \cup (w \cap \bar{y} \cap \bar{z})$$

The former is represented by

$$F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the second function, with y and z of the Karnaugh map in the same relative positions, is given by

$$F_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In order to perform the cap or cup operations for the series or parallel connection of the circuits, F_1 has to be expanded into a matrix for four variables. Such expansion yields

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which represents

$$f_1(w, x, y, z) = (y \cap z) \cup \bar{y}$$

The matrix for the series connection now becomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Since this is identical to F_2, it is demonstrated that the connection in series of the circuit of f_1 to that of f_2 leaves the overall transmission function unchanged. Similarly, the matrix for parallel connection

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which is identical to the expanded form of F_1, demonstrates that the transmission function $f_1 \cup f_2$ is equal to the transmission function f_1 alone.

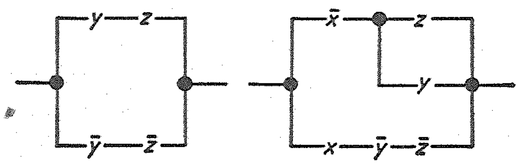

FIG. 9.2.3

As a second example, consider the circuits shown in figure 9.2.3. The matrices for these circuits are

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$F_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

with the Karnaugh maps as shown in figure 9.2.4.

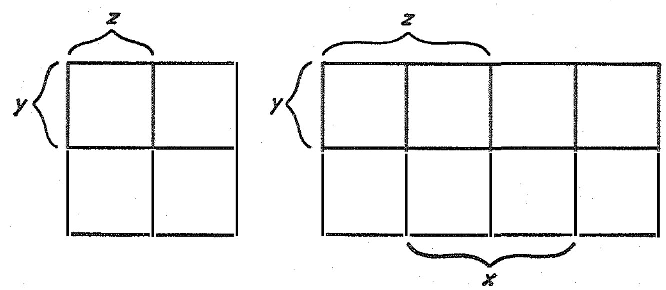

FIG. 9.2.4

Upon expansion to three variables, F_1 becomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The matrix for the series connection of the two circuits is therefore

$$F_1 \cap F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

yielding the circuit of figure 9.2.5.

The matrix for the parallel connection of the two circuits is

$$F_1 \cup F_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

yielding the circuit of figure 9.2.6.

The join of m matrices of the type just considered corresponds to matrix addition in the usual algebra of matrices with the Boolean operation of cup replacing the simple additive process. The meet of

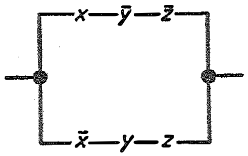

FIG. 9.2.5

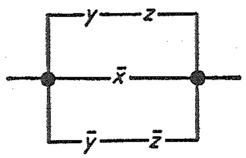

FIG. 9.2.6

m such matrices does not, however, correspond to matrix multiplication in the usual algebra of matrices, and also, unlike normal matrix multiplication, the operation is commutative.

A second way in which a Boolean matrix may arise can be seen if the following set of Boolean equations in two variables is considered:

$$z_1 \cup \bar{z}_2 = f_1$$
$$(z_1 \cap \bar{z}_2) \cup (\bar{z}_1 \cap z_2) = f_2$$

These can be expressed with the terms expanded into minimal polynomials as follows:

$$(1 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (0 \cap \bar{z}_1 \cap z_2) \cup (1 \cap z_1 \cap z_2) = f_1$$
$$(0 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (0 \cap z_1 \cap z_2) = f_2$$

Adopting a matrix representation similar to that for ordinary algebraic simultaneous equations yields

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

It can easily be seen that each row of the matrix of coefficients is the same as could be obtained by plotting the corresponding function on a 1×2^n Karnaugh map. The manipulation of matrix equations of this type has been described by Campeau,[1] and together with examples of their application will form the subject matter of chapter 10.

A third way of obtaining a Boolean matrix representing a switching circuit is to form a matrix $P = [p_{rs}]$, where each component represents the path between node r and node s of the circuit. For example, con-

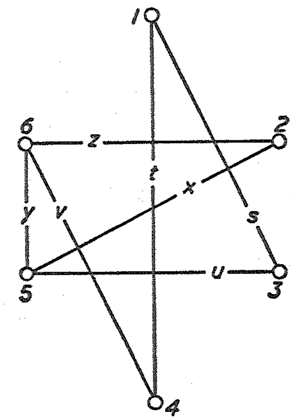

FIG. 9.2.7

sider the circuit of figure 9.2.7, which has been arranged with the nodes forming a regular polygon. The matrix, composed by the method just described, is

$$\begin{bmatrix} 1 & 0 & s & t & 0 & 0 \\ 0 & 1 & 0 & 0 & x & z \\ s & 0 & 1 & w & u & 0 \\ t & 0 & w & 1 & 0 & v \\ 0 & x & u & 0 & 1 & y \\ 0 & z & 0 & v & y & 1 \end{bmatrix}$$

It should be noticed that since the connection of a node with itself can always be regarded as a short circuit, all the p_{rs} for $r = s$ are unity. Also, whenever bi-directional devices such as relay switches are employed, p_{rs} will equal p_{sr}, i.e. the matrix will be symmetrical. It is clear that this will not be the case for electronic circuits in general.

It is desirable to distinguish between the matrix in which the components represent the circuit connection between pairs of nodes and the matrix in which they represent coefficients of minimal polynomials. The former type will therefore be termed *switching matrices*, and the latter will be termed *Boolean matrices*.

9.3 The Algebra of Switching Matrices

For the set of switching matrices to which $X = [x_{rs}]$, $Y = [y_{rs}]$, $Z = [z_{rs}], \ldots$ belong, the following definitions are postulated:

Equality:

$Y = Z$ if and only if $y_{rs} = z_{rs}$ for all r and s

Meet:

$Y \cap Z = [y_{rs} \cap z_{rs}]$

Join:

$Y \cup Z = [y_{rs} \cup z_{rs}]$

Complementation:

$\bar{Z} = [\bar{z}_{rs}]$ except for $r = s$

Inclusion:

$Y \leqq Z$ if and only if $y_{rs} \leqq z_{rs}$ for all r and s

Universal matrix, I:

The universal matrix has all components equal to unity

Zero matrix, $\emptyset$:

The zero matrix has components equal to zero for $r \neq s$, and equal to unity for $r = s$

Matrix product:

$YZ = \left[\bigcup_{k=1}^{m} y_{rk} \cap z_{ks}\right]$, where Y is of order $p \times m$ and Z is of order $m \times q$

Transpose:

$Z^t = [z_{sr}]$

Scalar product:

$x \cap Y = Z = [z_{rs}]$, where $z_{rs} = x \cap y_{rs}$ for $r \neq s$, and $z_{rs} = 1$ for $r = s$.

It is shown by Lunts[2] that with respect to the meanings of equality and inclusion and the operations of meet, join and complementation, switching matrices as here described constitute a Boolean algebra. If

a null matrix, 0, is included in which all components are zero and also the operation of forming the matrix product, then it is shown by Birkhoff[3] that the set of switching matrices forms a lattice-ordered semigroup with zero. The following theorems therefore follow from the definitions already postulated, and these should be compared with the theorems for switching algebra given in chapter 5:

$$
\begin{aligned}
&Z \cup \emptyset = Z\\
&Z \cap I = Z\\
&Z \cup I = I\\
&Z \cap \emptyset = \emptyset\\
&Z \cup Z = Z\\
&Z \cap Z = Z\\
&\bar{\bar{Z}} = Z\\
&Z \cup \bar{Z} = I\\
&Z \cap \bar{Z} = \emptyset\\
&Y \cup Z = Z \cup Y\\
&Y \cap Z = Z \cap Y\\
&Z \cup (Y \cap Z) = Z\\
&Z \cap (Y \cup Z) = Z\\
&Z \cup (Y \cap \bar{Z}) = Y \cup Z\\
&Z \cap (Y \cup \bar{Z}) = Y \cap Z\\
&X \cup Y \cup Z = X \cup (Y \cup Z) = (X \cup Y) \cup Z\\
&X \cap Y \cap Z = X \cap (Y \cap Z) = (X \cap Y) \cap Z\\
&(X \cap Y) \cup (X \cap Z) = X \cap (Y \cup Z)\\
&(X \cup Y) \cap (X \cup Z) = X \cup (Y \cap Z)\\
&(X \cap Y) \cup (Y \cap Z) \cup (Z \cap \bar{X}) = (X \cap Y) \cup (Z \cap \bar{X})\\
&(X \cup Y) \cap (Y \cup Z) \cap (Z \cup \bar{X}) = (X \cup Y) \cap (Z \cup \bar{X})\\
&(X \cup Y) \cap (\bar{X} \cup Z) = (X \cap Z) \cup (\bar{X} \cap Y)\\
&\overline{(Y \cup Z)} = \bar{Y} \cap \bar{Z}\\
&\overline{(Y \cap Z)} = \bar{Y} \cup \bar{Z}\\
&\text{In general } YZ \neq ZY\\
&X(Y \cup Z) = XY \cup XZ\\
&X(Y \cap Z) \leqq XY \cap XZ\\
&(X \cup Y)Z = XZ \cup YZ\\
&(X \cap Y)Z \leqq XZ \cap YZ\\
&X(YZ) = (XY)Z\\
&0 \cup Z = Z\\
&0 \cap Z = 0Z = Z0 = 0
\end{aligned}
$$

$$\emptyset Z = Z\emptyset = Z$$
$$(Y \cup Z)^t = Y^t \cup Z^t$$
$$(Y \cap Z)^t = Y^t \cap Z^t$$
$$(YZ)^t = Z^t Y^t$$
$$(Z^t)^t = Z$$
$$II = I$$
$$ZI = IZ = I$$
$$\overline{(Z^t)} = \bar{Z}^t$$
$$\emptyset\emptyset = \emptyset$$
$$(Z^p)^q = Z^{pq}$$
$$Z^p Z^q = Z^{p+q}$$

It will be noticed that approximately the first half of the above theorems apply to switching matrices because these constitute a Boolean algebra. The remainder follow from the introduction of a null matrix and the operation of Boolean matrix multiplication. The following order relations must also apply:

$\emptyset \leqq Z \leqq I$ for all Z
$Z \leqq Z$
$Y \leqq Z$ and $Z \leqq Y$ if and only if $Y = Z$
$X \leqq Y$ and $Y \leqq Z$ implies $X \leqq Z$
$Y \leqq Z$ if and only if $Y \cup Z = Z$
$Y \leqq Z$ if and only if $Y \cap Z = Y$
$X \leqq Y$ implies $XZ \leqq YZ$ and $ZX \leqq ZY$ for all Z
$0 \leqq Z \leqq I$ for all Z

The formal definition of symmetry may be carried over direct from ordinary matrix theory. Thus, a switching matrix is said to be symmetric if

$$Z^t \cap \bar{Z} = \emptyset$$

However, if the matrix is skew-symmetric then

$$Z^t \cap Z = \emptyset$$

Symmetry is invariant under the operations of meet, join and complementation, but skew-symmetry is invariant under the meet operation only. Any switching matrix can be uniquely decomposed into a join of a symmetric matrix and a skew-symmetric matrix.

If the inverse of a matrix, where it exists, is denoted by Z^{-1} then the matrix is said to be *orthogonal* if

$$Z^{-1} = Z^t$$

A switching matrix has an inverse if and only if it is orthogonal. Further, if a switching matrix is both symmetric and orthogonal then it immediately follows that it is involutory, that is

$$ZZ = Z^2 = \emptyset$$

One further useful theorem, which is due to Lunts,[2] states that if Z is any switching matrix of order m, then there exists a positive integer $q \leqq m-1$, such that

$$Z \leqq Z^2 \leqq \ldots \leqq Z^q = Z^{q+1} = \ldots$$

Using the theorems listed above, it is now possible to consider the application of the algebra of switching matrices to problems of circuitry.

9.4 Types of Switching Matrices

Consider first the circuit having n inputs, $x, y, z, \ldots$ and t outputs in which the state of the connection between any two output terminals depends only upon the values of all or some of the input variables. This can be written

$$f_{rs} = f_{rs}(x, y, z, \ldots)$$

Since

$$f_{rs} = 1 \text{ for } r = s$$

the t^2 functions can be used as components of a $t \times t$ switching matrix, termed the *output matrix* of the given circuit. As an example, consider the circuit shown in figure 9.4.1. The output matrix is obtained by

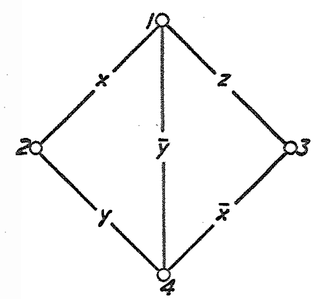

FIG. 9.4.1

considering all possible paths between every pair of nodes, and, after minimization of each individual function comprising an element of the matrix, is

$$F = \begin{bmatrix} 1 & x\cup(y\cap z) & z\cup(\bar{x}\cap\bar{y}) & x\cup\bar{y}\cup z \\ x\cup(y\cap z) & 1 & (x\cap z)\cup(\bar{x}\cap y) & x\cup y \\ z\cup(\bar{x}\cap\bar{y}) & (x\cap z)\cup(\bar{x}\cap y) & 1 & \bar{x}\cup y\cup z \\ x\cup\bar{y}\cup z & x\cup y & \bar{x}\cup y\cup z & 1 \end{bmatrix}$$

Circuits with the same output matrix are termed *equivalent*.

Next, within a given circuit a number of non-terminal nodes, $t+1$, $t+2, \ldots,$ can be so chosen that between any two nodes there appears at most a single element-state or a group of single element-states in

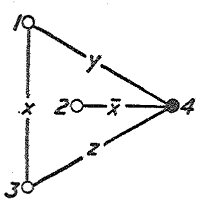

FIG. 9.4.2

parallel, and also so that every element in the network is included in the connection between some pair of nodes. If there are k non-terminal nodes, then a matrix, P, is obtained having order $t+k$ and in which p_{rs} has the value 0 if there is no direct connection between node r and node s, the value 1 if there is a short circuit between nodes r and s, or represents a single element-state or a join of element-states. Such a matrix, P, is termed a *primitive connection matrix*.

As an example, consider the circuit shown in figure 9.4.2. In this circuit node 4 is non-terminal. This is indicated by the small circle representing the node being filled in, whereas the terminal nodes are shown open. The primitive connection matrix is

$$P = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & \bar{x} \\ x & 0 & 1 & z \\ y & \bar{x} & z & 1 \end{bmatrix}$$

whereas the output matrix can be seen to be

$$F = \begin{bmatrix} 1 & \bar{x}\cap y & x\cup(y\cap z) \\ \bar{x}\cap y & 1 & \bar{x}\cap z \\ x\cup(y\cap z) & \bar{x}\cap z & 1 \end{bmatrix}$$

Yet a third type of switching matrix can be envisaged, namely one in which the components are switching functions connecting pairs of nodes, terminal and non-terminal, but where the number of non-

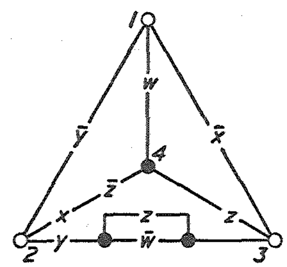

FIG. 9.4.3

terminal nodes may be insufficient to give a primitive connection matrix, though all elements are accounted for. Such a matrix has been described by Semon[4] simply as a *connection matrix*.

As an example, consider the circuit of figure 9.4.3. The connection matrix for this circuit is

$$C = \begin{bmatrix} 1 & \bar{y} & \bar{x} & w \\ \bar{y} & 1 & y\cap(\bar{w}\cup z) & x\cap\bar{z} \\ \bar{x} & y\cap(\bar{w}\cup z) & 1 & z \\ w & x\cap\bar{z} & z & 1 \end{bmatrix}$$

9.5 The Y-Δ Transformation

As in electrical network theory of linear networks, it is possible to carry out what is in effect a Y-Δ transformation. Figure 9.5.1 illustrates this transformation.

The two circuits have the same output matrix, namely

$$F = \begin{bmatrix} 1 & x\cap y & x\cap z \\ x\cap y & 1 & y\cap z \\ x\cap z & y\cap z & 1 \end{bmatrix}$$

and hence are equivalent. It should be noticed that the transformation has removed the non-terminal node present in the Y-circuit. The transformation from the Δ-circuit to the Y-circuit is the dual of that from Y to Δ.

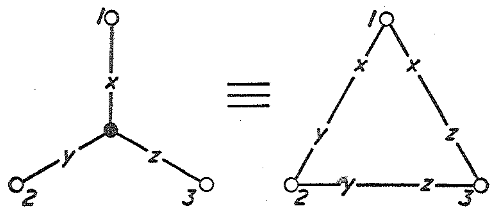

FIG. 9.5.1

The Y-Δ transformation may be extended to junction points with more than three legs, in which case it is usually known as the star-mesh transformation. An example with four legs is shown in figure 9.5.2.

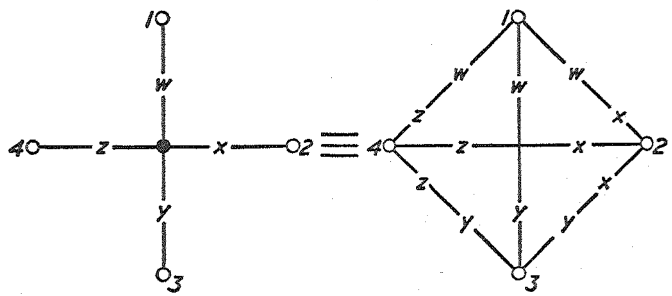

FIG. 9.5.2

The two circuits are equivalent, their common output matrix being

$$F = \begin{bmatrix} 1 & w\cap x & w\cap y & w\cap z \\ w\cap x & 1 & x\cap y & x\cap z \\ w\cap y & x\cap y & 1 & y\cap z \\ w\cap z & x\cap z & y\cap z & 1 \end{bmatrix}$$

Again, the transformation from mesh to star is the dual of that from star to mesh, but in certain circumstances it is not possible to effect this transformation.

9.6 Analysis using Switching Matrices

In section 9.5 it was shown that the Y-Δ or star-mesh transformation can dispose of a node which is non-terminal. By formalizing this transformation it is possible to reduce a given connection matrix to one representing a circuit in which no non-terminal nodes remain.

The procedure in the case of some non-terminal node u is to replace each component c_{rs} of the connection matrix C by its join with the

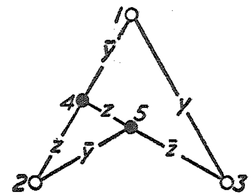

FIG. 9.6.1

meet of the component c_{ru} in row r and column u of C and the component c_{us} in row u and column s of C. Row and column u are then deleted. This process is then repeated until no non-terminal nodes remain. The resulting $t\times t$ connection matrix is termed a *reduced connection matrix* and is denoted by C_0. The matrix obtained from C by the removel of node u only is denoted by C_{-u} unless it is itself the reduced connection matrix.

As an example consider the circuit shown in figure 9.6.1. The primitive connection matrix is

$$P = \begin{bmatrix} 1 & 0 & y & \bar{y} & 0 \\ 0 & 1 & 0 & z & \bar{y} \\ y & 0 & 1 & 0 & \bar{z} \\ \bar{y} & z & 0 & 1 & z \\ 0 & \bar{y} & \bar{z} & z & 1 \end{bmatrix}$$

Row and column 5 can be eliminated by replacing each p_{rs} by

$$p_{rs}\cup(p_{r5}\cap p_{5s})$$

Thus, $p_{12}=0$ is replaced by $p_{12}\cup(p_{15}\cap p_{52}) = 0\cup(0\cap\bar{y}) = 0$, i.e. in this instance it remains unchanged. Similarly, it will be found that p_{13}, p_{14}, and p_{21} are unchanged. However, $p_{23} = 0$ is replaced by $0\cup(\bar{y}\cap\bar{z}) = (\bar{y}\cap\bar{z})$ and p_{32} must also be replaced in the same way.

Clearly, since the join of 1 with any function is 1, all p_{rs} with $r = s$ will remain unchanged. When the process has been completed and row and column 5 deleted, there remains the matrix

$$C_{-5} = \begin{bmatrix} 1 & 0 & y & \bar{y} \\ 0 & 1 & \bar{y}\cap\bar{z} & z \\ y & \bar{y}\cap\bar{z} & 1 & 0 \\ \bar{y} & z & 0 & 1 \end{bmatrix}$$

Reference to figure 9.6.1 shows that this is indeed the connection matrix for the circuit with node 5 removed. The effect of such removal is to permit a direct path between nodes 2 and 3 represented by $(\bar{y}\cap\bar{z})$

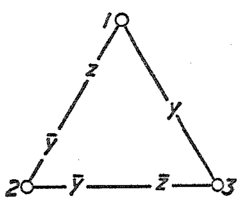

FIG. 9.6.2

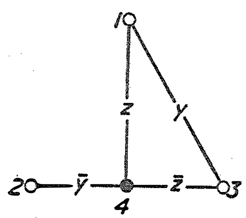

FIG. 9.6.3

but the other possible new direct path, that between nodes 3 and 4, remains 0 since it is not possible to transmit through an element-state and its complement.

The process is now continued in its next stage in order to eliminate row and column 4. Since no non-terminal nodes remain, the resulting matrix is also the reduced connection matrix C_0. This is

$$\begin{aligned} C_{-5,4} &= \begin{bmatrix} 1 & \bar{y}\cap z & y \\ \bar{y}\cap z & 1 & \bar{y}\cap\bar{z} \\ y & \bar{y}\cap\bar{z} & 1 \end{bmatrix} \\ &= C_0 \end{aligned}$$

C_0 represents the circuit shown in figure 9.6.2 which has no non-terminal nodes.

If figure 9.6.2 is investigated closely, it will be apparent that it could equally well have resulted from removing the node 4 from the circuit shown in figure 9.6.3. The primitive connection matrix for this circuit can be obtained directly from C_0 above, the common factor

$\bar{y}$ of $(\bar{y}\cap z)$ and $(\bar{y}\cap\bar{z})$ giving the clue to the procedure. The matrix is

$$P = \begin{bmatrix} 1 & 0 & y & z \\ 0 & 1 & 0 & \bar{y} \\ y & 0 & 1 & \bar{z} \\ z & \bar{y} & \bar{z} & 1 \end{bmatrix}$$

It is thus possible to perform the operation of node-insertion directly upon a suitable matrix.

The main problem of analysis is to obtain the corresponding output matrix from a given connection matrix. The relationship was first derived by Lunts[2] and later generalized by Hohn and Schissler;[5] it may be stated thus:

> If C is any connection matrix of a t-terminal circuit, C_0 the corresponding reduced connection matrix, and F the output matrix, then there exists an integer k between 0 and t such that $C_0^{t-k} = F$.

Reference to the matrix C_0 for the circuit of figure 9.6.2, for example, shows that it is in fact the output matrix of the circuit since $C_0 = C_0^2$.

The generalized form of the relationship as stated above can be justified by reference to the theorem due to Lunts stated at the end of section 9.3. From this theorem it is seen that there exists an integer k between 0 and t such that $C_0^{t-k} = C_0^{t-k+1} = \ldots$; it is therefore only necessary to demonstrate that $C_0^{t-1} = F$. If the components of C_0 are denoted by c_{rs}, then the component of C_0^2 in row r and column s is

$$(c_{r1}\cap c_{1s})\cup(c_{r2}\cap c_{2s})\cup\ldots\cup(c_{rt}\cap c_{ts})$$

This function is equal to unity for $r \neq s$ only if the input variables are in a condition such that there is a direct path from node r to node s or else a path via one intermediate node. In a similar manner, the component of C_0^3 in row r and column s is unity only when there is either a direct path from node r to node s, or a path via one intermediate node or via two intermediate nodes. Since no path can require more than $t-2$ intermediate nodes, the component of C_0^{t-1} in row r and column s is unity only when the input variables are in a condition which interconnects nodes r and s. Hence

$$C_0^{t-1} = F$$

It now immediately follows that the reduced connection matrix of a two-terminal circuit is the output matrix, and also that the component of an output matrix F in row r and column s may be determined by considering a circuit as a two-terminal circuit connecting nodes r and s with all other nodes removed. Further, a necessary and sufficient condition for a symmetric switching matrix C to be an output matrix is that $C^2 = C$.

It is sometimes possible to detect and remove redundant elements directly from a switching matrix. As an example consider a case where two identical elements appear in the same row together with a third identical element in one of the same columns. Such is the case in the matrix

$$\begin{bmatrix} 1 & y \cup z & z \\ y \cup z & 1 & x \cup z \\ z & x \cup z & 1 \end{bmatrix}$$

The term z in component 12 is redundant as there is a path from node 1 to node 2 via node 3 whenever z is operated. The term can therefore be omitted together with the z in component 21. Alternatively, one of the z's appearing in components 13 and 23 could be omitted together with its symmetrical appearance.

As a second example, consider the matrix

$$\begin{bmatrix} 1 & x \cup y & \bar{x} \cup (y \cap z) \\ x \cup y & 1 & z \\ \bar{x} \cup (y \cap z) & z & 1 \end{bmatrix}$$

Here the factors of the term $(y \cap z)$ appearing in row 1 column 3 also appear elsewhere in row 1 and column 3. There is thus a path from node 1 to node 3 via node 2 whenever both y and z are operated, and hence the term $(y \cap z)$ can be omitted from components 13 and 31.

Again, consider the matrix

$$\begin{bmatrix} 1 & y & x \cup (\bar{y} \cap z) \\ y & 1 & y \\ x \cup (\bar{y} \cap z) & y & 1 \end{bmatrix}$$

Here the element-state $\bar{y}$ in component 13 is redundant since there is a path from node 1 to node 3 whenever z is operated regardless of the state of y. In effect, removal of an element-state is equivalent to replacing it by 0 in the parallel case and by 1 in the series case. This follows simply from the two theorems

$$z \cup 0 = z$$

and

$$z \cap 1 = z$$

It is important that this process should be carried out in individual successive steps since each operation may alter the conditions for redundancy of other terms.

In addition to the process just described, it is frequently useful to replace elements of a connection matrix by switching functions other than 0 and 1. However, before considering the conditions under which given components can be replaced without changing the overall output, it is necessary to examine the significance of the determinant of a switching matrix.

9.7 The Determinant of a Switching Matrix

The determinant of a switching matrix Z with components z_{rs} is defined as the join of the $m!$ meets of the components of Z in which each row subscript and each column subscript is represented once only in each meet. For example, if

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

then

$$\det Z = (z_{11} \cap z_{22}) \cup (z_{12} \cap z_{21})$$

The coefficients of $z_{r1}, z_{r2}, \ldots, z_{rm}$ may be collected, giving

$$\det Z = (Z_{r1} \cap z_{r1}) \cup (Z_{r2} \cap z_{r2}) \cup \ldots \cup (Z_{rm} \cap z_{rm})$$

Each term of $\det Z$ involves each row and each column only once; hence Z_{rs} is obtained from Z by deleting row r and column s. Thus an expansion by minors exists analogous to that for ordinary determinants.

In addition to the expansion by minors, a number of other properties of ordinary determinants hold good for the determinant of a switching matrix. Two rows or columns may be interchanged without changing the value of a determinant, but in general the term by term join of two rows or columns will change its overall value. The determinant of the transpose of a matrix is equal to the determinant of the original matrix. If Z is formed from Y by the meet of each component of a row or column of Y with an element x, then

$$\det Z = x \cap \det Y$$

The Laplacian expansion by minors of a selected set of rows or columns is also valid. On the other hand, the fact that Z has two identical rows or columns does not imply that $\det Z = 0$.

Consider now the connection matrix C representing some switching network. If the row and column of a component c_{rs} are deleted, then the determinant of the contracted matrix so obtained is the switching function representing an output taken across nodes r and s; that is to say, it is the component f_{rs} of the output matrix F of the circuit. As an example, the connection matrix for the switching network of figure 9.4.1 is

$$C = \begin{bmatrix} 1 & x & z & \bar{y} \\ x & 1 & 0 & y \\ z & 0 & 1 & \bar{x} \\ \bar{y} & y & \bar{x} & 1 \end{bmatrix}$$

Deletion of row 1 and column 2 yields

$$\begin{bmatrix} x & 0 & y \\ z & 1 & \bar{x} \\ \bar{y} & \bar{x} & 1 \end{bmatrix}$$

the determinant of which is equal to

$$[x \cap (1 \cup \bar{x})] \cup \{y \cap [(\bar{x} \cap z) \cup \bar{y}]\} = x \cup (y \cap z)$$

which is the component f_{12} of the output matrix F of the circuit as given in section 9.4. Deletion of row 2 and column 3 yields

$$\begin{bmatrix} 1 & x & \bar{y} \\ z & 0 & \bar{x} \\ \bar{y} & y & 1 \end{bmatrix}$$

the determinant of which is equal to

$$\{1 \cap [0 \cup (\bar{x} \cap y)]\} \cup \{x \cap [(\bar{x} \cap \bar{y}) \cup z]\} \cup \{\bar{y} \cap [(y \cap z) \cup 0]\}$$

which in turn simplifies to

$$(\bar{x} \cap y) \cup (x \cap z)$$

the component f_{23} of the output matrix. A detailed analytical proof of this work has been given by Semon.[4]

It is possible to extend the connection matrix of a circuit to include components representing particular outputs and also to include a power supply. As an example, consider the circuit of figure 9.7.1.

The connection matrix is

$$C = \begin{bmatrix} 1 & x & y & \bar{y} & E- \\ x & 1 & y & 0 & 0 \\ y & y & 1 & z & F_1 \\ \bar{y} & 0 & z & 1 & F_2 \\ E+ & 0 & F_1 & F_2 & 1 \end{bmatrix}$$

It should be noticed that the supply direction has been allowed for in that $c_{15} \neq c_{51}$. To obtain the switching function for a particular output it is necessary to cross out the row and column containing the

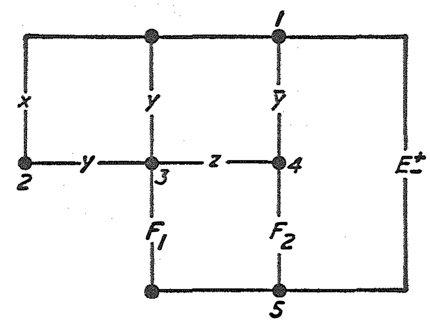

FIG. 9.7.1

positive sense of the generator and the row and column containing the required output F_i. The determinant of the resulting minor is the required switching expression for F_i. Thus, in the example being considered, the switching expression for F_1 is

$$\det \begin{bmatrix} x & y & \bar{y} \\ 1 & y & 0 \\ 0 & z & 1 \end{bmatrix}$$

After expansion and simplification, this is seen to be

$$y \cup z$$

Similarly, the expression for F_2 is

$$\det \begin{bmatrix} x & y & \bar{y} \\ 1 & y & 0 \\ y & 1 & z \end{bmatrix}$$

which simplifies to

$$\bar{y} \cup z$$

9.8 Further Analysis using Switching Matrices

Section 9.7 concluded with a reference to replacing the components of a connection matrix by switching functions other than 0 and 1. It is now possible to obtain the conditions for such a substitution which leaves the overall output unchanged.

Suppose that component c_{rs} of C is to be replaced. It is possible to evaluate the component F or f_{rs} by the method described in section 9.7, expressing it in the form $(a \cap z) \cup b$. Clearly, for the output to remain unchanged

$$(a \cap z) \cup b = (a \cap c_{rs}) \cup b$$

This will be so if and only if both

$$a \cap \bar{b} \cap \bar{c}_{rs} \cap z = 0$$

and

$$a \cap \bar{b} \cap c_{rs} \cap \bar{z} = 0$$

These two conditions imply

$$(a \cap \bar{b} \cap c_{rs}) \leqq z \leqq (\bar{a} \cup b \cup c_{rs})$$

Hence, provided that z satisfies this order relation, it may replace the component c_{rs}.

There are some special cases which are of sufficient interest to be considered separately.

First, if $a \leqq b$ then $a \cap \bar{b} = 0$ and $\bar{a} \cup b = 1$. Hence any z whatsoever may be substituted for c_{rs} without altering F.

Secondly, if $a = (\bar{z} \cap a_0) \cup (z \cap a_1)$ and $b = (\bar{z} \cap b_0) \cup (z \cap b_1)$, then c_{rs} may be replaced by 1 for $a_0 \leqq b_0$ and by 0 for $a_1 \leqq b_1$.

Thirdly, if $c_{rs} = 0$, then $b = f_{rs}$ and the order relation for z becomes

$$0 \leqq z \leqq (\bar{a} \cup b)$$

Fourthly, if $c_{rs} = 1$, then the order relation becomes

$$(a \cap \bar{b}) \leqq z \leqq 1$$

For a first example, consider the circuit illustrated in figure 9.8.1. The connection matrix for this circuit is

$$C = \begin{bmatrix} 1 & y & 0 & 0 & E- \\ y & 1 & \bar{y} & 0 & 0 \\ 0 & \bar{y} & 1 & x & 0 \\ 0 & 0 & x & 1 & F \\ E+ & 0 & 0 & F & 1 \end{bmatrix}$$

By striking out the row and column containing F and the row and column containing $E+$, the value of F is obtained as

$$\det \begin{bmatrix} y & 0 & 0 \\ 1 & \bar{y} & 0 \\ \bar{y} & 1 & x \end{bmatrix}$$

Replacing the components $\bar{y}$ by z yields

$$\det \begin{bmatrix} y & 0 & 0 \\ 1 & z & 0 \\ z & 1 & x \end{bmatrix} = x \cap y \cap z$$

Thus $a = x \cap y$ and $b = 0$, and any z can be substituted provided that it satisfies the order relation

$$(x \cap y \cap 1 \cap \bar{y}) \leqq z \leqq (\bar{x} \cup \bar{y} \cup 0 \cup \bar{y})$$

that is

$$0 \leqq z \leqq (\bar{x} \cup \bar{y})$$

Hence $\bar{y}$ may be replaced by an open circuit as figure 9.8.1 clearly shows.

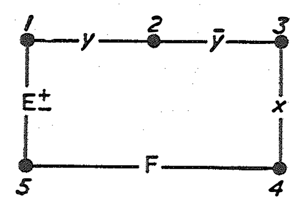

FIG. 9.8.1

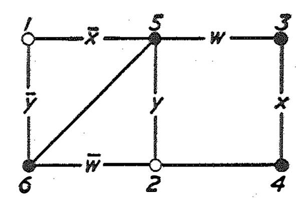

FIG. 9.8.2

Next, consider the circuit of figure 9.8.2. The connection matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \bar{x} & \bar{y} \\ 0 & 1 & 0 & 1 & y & \bar{w} \\ 0 & 0 & 1 & x & w & 0 \\ 0 & 1 & x & 1 & 0 & 0 \\ \bar{x} & y & w & 0 & 1 & 1 \\ \bar{y} & \bar{w} & 0 & 0 & 1 & 1 \end{bmatrix}$$

It should be noted here that both the power source and a component representing the output have been omitted in the circuit illustration and in the matrix. Clearly, the output is to be taken across nodes 1 and 2. Striking out row 2 and column 1 yields

$$\begin{bmatrix} 0 & 0 & 0 & \bar{x} & \bar{y} \\ 0 & 1 & x & w & 0 \\ 1 & x & 1 & 0 & 0 \\ y & w & 0 & 1 & 1 \\ \bar{w} & 0 & 0 & 1 & 1 \end{bmatrix}$$

the determinant of which represents the output. This simplifies to

$$[\bar{x} \cap (y \cup \bar{w})] \cup (x \cap \bar{y})$$

If z is now substituted for the switching function between nodes 1 and 5, the output becomes

$$[(x \cup y \cup \bar{w}) \cap z] \cup [\bar{y} \cap (x \cup \bar{w})]$$

Hence

$$a = (x \cup y \cup \bar{w})$$

and

$$b = \bar{y} \cap (x \cup \bar{w})$$

and the condition for substitution is given by the order relation

$$(\bar{x} \cap y) \leqq z \leqq (\bar{x} \cup \bar{y})$$

It is therefore permissible to substitute z provided that it is equal either to $(\bar{x} \cap y)$ or to $(\bar{x} \cup \bar{y})$, since the only other term within the limits given is $\bar{x}$, the original c_{15} for which z was substituted.

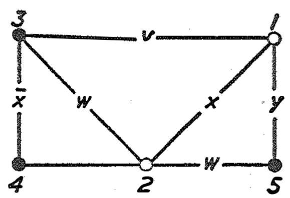

FIG. 9.8.3

For a third example, consider the circuit of figure 9.8.3. The connection matrix in this case is

$$C = \begin{bmatrix} 1 & x & v & 0 & y \\ x & 1 & w & 1 & w \\ v & w & 1 & \bar{x} & 0 \\ 0 & 1 & \bar{x} & 1 & 0 \\ y & w & 0 & 0 & 1 \end{bmatrix}$$

giving for the output between nodes 1 and 2

$$\det \begin{bmatrix} x & v & 0 & y \\ w & 1 & \bar{x} & 0 \\ 1 & \bar{x} & 1 & 0 \\ w & 0 & 0 & 1 \end{bmatrix}$$

If z is now inserted into the short-circuit between nodes 2 and 4, the output becomes

$$\det \begin{bmatrix} x & v & 0 & y \\ w & 1 & \bar{x} & 0 \\ z & \bar{x} & 1 & 0 \\ w & 0 & 0 & 1 \end{bmatrix}$$

which, when expanded and simplified, becomes

$$[(v \cap \bar{x}) \cap z] \cup [x \cup (v \cap w) \cup (w \cap y)]$$

Thus

$$a = v \cap \bar{x}$$

and

$$b = [x \cup (v \cap w) \cup (w \cap y)]$$

Since substitution has been made for a short-circuit, this is one of the special cases. The order relation for z is

$$(a \cap \bar{b}) \leqq z \leqq 1$$

giving, after substitution and simplification,

$$(v \cap \bar{w} \cap \bar{x}) \leqq z \leqq 1$$

Thus the short-circuit between terminals 2 and 4 may be replaced by any of the 127 functions which contain $(v \cap \bar{w} \cap \bar{x})$, the output between nodes 1 and 2 being unaltered.

If a minimized circuit is considered, then it is clear that it is not possible to replace a component not already 0 or 1 by an open- or a short-circuit without altering the output. Were it so possible, then a network would have been produced with fewer switching elements and the original could not have been minimal. It is not true, however, that a circuit for which replacement limits of 0 and 1 do not occur is necessarily minimal.

It is possible to break up a network into sub-circuits by direct operation upon its connection matrix. For example, consider the network shown in figure 9.8.4. Here no attempt has been made to distinguish between terminal and non-terminal nodes, since such distinction is not relevant to the present problem.

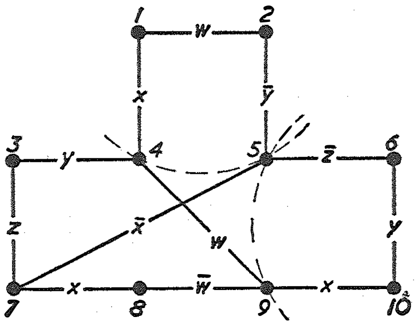

FIG. 9.8.4

The connection matrix is

$$C = \begin{bmatrix} 1 & w & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 1 & 0 & 0 & \bar{y} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 & z & 0 & 0 & 0 \\ x & 0 & y & 1 & 0 & 0 & 0 & 0 & w & 0 \\ 0 & \bar{y} & 0 & 0 & 1 & \bar{z} & \bar{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{z} & 1 & 0 & 0 & 0 & y \\ 0 & 0 & z & 0 & \bar{x} & 0 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 1 & \bar{w} & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & \bar{w} & 1 & x \\ 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & x & 1 \end{bmatrix}$$

Suppose that it is desired to disconnect the sub-circuits 4125 and 56109 as indicated in the figure. Then, each row and column for each of the

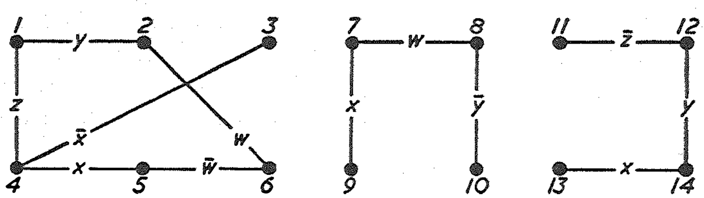

FIG. 9.8.5

nodes thus divided must be replaced by two or three rows and columns, according to the number of separate sub-circuits containing such nodes. Rows and columns 4 and 9 are thus each replaced by two rows and columns, and row and column 5 are replaced by three rows and columns. The resultant matrix is

$$C = \begin{bmatrix} 1 & w & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 1 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 \\ x & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{x} & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{z} & 1 & 0 & 0 & 0 & 0 & \bar{y} \\ 0 & 0 & z & 0 & 0 & \bar{x} & 0 & 0 & 0 & 1 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 1 & \bar{w} & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & \bar{w} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{y} & 0 & 0 & 0 & x & 1 \end{bmatrix}$$

The nodes are now renumbered to give three separate circuits shown in figure 9.8.5.

The rearranged matrix now becomes

$$\left[\begin{array}{cccccc|cccc|cccc} 1 & y & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 1 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \bar{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z & 0 & \bar{x} & 1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 1 & \bar{w} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & \bar{w} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & w & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & 1 & 0 & \bar{y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{y} & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{z} & 1 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & x & 1 \end{array}\right]$$

which is of the form $\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$

where A, B, C are the connection matrices of the three sub-circuits, and 0 denotes a rectangular null matrix.

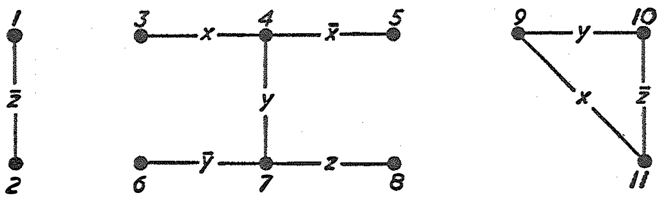

FIG. 9.8.6

This immediately suggests the possibility of the reverse process. Consider, therefore, the circuits of figure 9.8.6. The matrix for the overall circuit with the sub-circuits disconnected is

$$\left[\begin{array}{cc|cccccc|ccc}
1 & \bar{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{z} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 1 & \bar{x} & 0 & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{x} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \bar{y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y & 0 & \bar{y} & 1 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & y & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 1 & \bar{z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & \bar{z} & 1
\end{array}\right]$$

If it is now desired to join up the sub-circuits by terminals 1–3, 2–6, 4–9, 5–10, and 8–11, all that is necessary is to contract the matrix, replacing row 1 by the join of row 1 and row 3, and column 1 by the join of column 1 and column 3, and so on. This yields the new matrix

$$\begin{bmatrix}
1 & \bar{z} & x & 0 & 0 & 0 \\
\bar{z} & 1 & 0 & 0 & \bar{y} & 0 \\
x & 0 & 1 & \bar{x}\cup y & y & x \\
0 & 0 & \bar{x}\cup y & 1 & 0 & \bar{z} \\
0 & \bar{y} & y & 0 & 1 & z \\
0 & 0 & x & \bar{z} & z & 1
\end{bmatrix}$$

which represents the overall circuit with sub-circuits connected up and terminals renumbered as shown in figure 9.8.7.

An alternative method of obtaining a matrix when the sub-circuits are connected up has been suggested by Semon.[4] In this method, in order to connect any node r to any node s, where r and s are in different

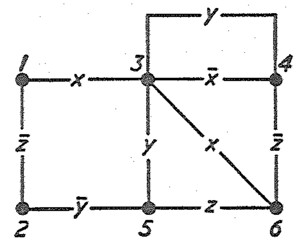

FIG. 9.8.7

sub-circuits, the zero components c_{rs} and c_{sr} are replaced by 1. This retains a matrix of the same order as the original matrix for the disconnected circuits, and a contraction process has to be applied. The method suggested by the author,[6] already described, leads immediately to a matrix of order $m-k$, where m is the order of the disconnected circuit matrix and k is the number of terminals absorbed when the sub-circuits are connected up.

9.9 Synthesis using Switching Matrices

If the required output of an n-terminal circuit is given in the form of a truth table, then the output matrix F may readily be obtained. For example, consider the circuit specified by table 9.9.1.

y	z	f_{12}	f_{13}	f_{14}	f_{23}	f_{24}	f_{34}
0	0	1	0	1	0	1	0
0	1	0	1	0	0	0	0
1	0	0	0	1	1	0	0
1	1	0	0	0	1	0	0

TABLE 9.9.1

First, it is necessary to test the required conditions for consistency. This is done, row by row, by checking that wherever a path is indicated both from node r to node s and from node s to node t, there is also a path indicated from node r to node t. That is to say, if

$$f_{rs} = f_{st} = 1$$

then

$$f_{rt} = 1$$

Thus in the first row of table 9.9.1, $f_{12} = f_{24} = 1$; it is therefore necessary to check that $f_{14} = 1$. It will readily be found that all the rows of the table given are consistent. The output matrix may now be written down since, from the various columns of the table,

$$\begin{aligned}
f_{12} &= \bar{y} \cap \bar{z} \\
f_{13} &= \bar{y} \cap z \\
f_{14} &= (\bar{y} \cap \bar{z}) \cup (y \cap \bar{z}) = \bar{z} \\
f_{23} &= (y \cap \bar{z}) \cup (y \cap z) = y \\
f_{24} &= \bar{y} \cap \bar{z} \\
f_{34} &= 0
\end{aligned}$$

Hence

$$F = \begin{bmatrix} 1 & \bar{y} \cap \bar{z} & \bar{y} \cap z & \bar{z} \\ \bar{y} \cap \bar{z} & 1 & y & \bar{y} \cap \bar{z} \\ \bar{y} \cap z & y & 1 & 0 \\ \bar{z} & \bar{y} \cap \bar{z} & 0 & 1 \end{bmatrix}$$

Since there is a path from node 1 to node 4 if $\bar{z} = 1$, and from node 4 to node 2 if both $\bar{y}$ and $\bar{z} = 1$, the terms f_{12} and f_{21} are redundant and may be replaced by 0 representing an open circuit between nodes 1 and 2 direct. The matrix now becomes

$$\begin{bmatrix} 1 & 0 & \bar{y} \cap z & \bar{z} \\ 0 & 1 & y & \bar{y} \cap \bar{z} \\ \bar{y} \cap z & y & 1 & 0 \\ \bar{z} & \bar{y} \cap \bar{z} & 0 & 1 \end{bmatrix}$$

This represents the network shown in figure 9.9.1.

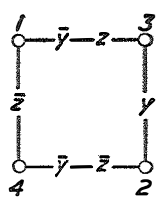

FIG. 9.9.1

For a two-terminal circuit, the synthesis problem consists of starting with an output matrix

$$F = \begin{bmatrix} 1 & f \\ f & 1 \end{bmatrix}$$

and successively inserting nodes to reduce the matrix components to simple form so that ultimately a primitive connection matrix is obtained. It is not in fact necessary to write down a new matrix at each stage; the whole process can be performed by extending the original output matrix, as the following example shows.

Let $$f = (\bar{x} \cap \bar{y}) \cup (x \cap y) \cup (\bar{y} \cap \bar{z})$$

The output matrix in its original form is therefore

$$F = \begin{bmatrix} 1 & (\bar{x} \cap \bar{y}) \cup (x \cap y) \cup (\bar{y} \cap \bar{z}) \\ (\bar{x} \cap \bar{y}) \cup (x \cap y) \cup (\bar{y} \cap \bar{z}) & 1 \end{bmatrix}$$

The term $(x \cap y)$ can be removed from f_{12} and f_{21} by inserting a third node, yielding the matrix

$$\begin{bmatrix} 1 & (\bar{x} \cap \bar{y}) \cup (\bar{y} \cap \bar{z}) & x \\ (\bar{x} \cap \bar{y}) \cup (\bar{y} \cap \bar{z}) & 1 & y \\ x & y & 1 \end{bmatrix}$$

$(\bar{x} \cap \bar{y}) \cup (\bar{y} \cap \bar{z})$ now simplifies to $\bar{y} \cap (\bar{x} \cup \bar{z})$, and a fourth node is inserted to yield

$$\begin{bmatrix} 1 & 0 & x & \bar{y} \\ 0 & 1 & y & \bar{x} \cup \bar{z} \\ x & y & 1 & 0 \\ \bar{y} & \bar{x} \cup \bar{z} & 0 & 1 \end{bmatrix}$$

The resultant circuit is shown in figure 9.9.2.

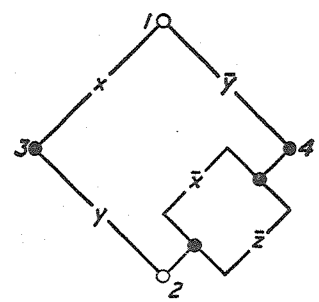

FIG. 9.9.2

Instead of writing down a separate matrix for each node insertion, all the stages may be represented by extending one matrix, yielding

$$\begin{bmatrix} 1 & (\bar{x}\cap\bar{y})\cup(x\cap y)\cup(\bar{y}\cap\bar{z}) & x & \bar{y} \\ 0 & 1 & y & \bar{x}\cup\bar{z} \\ x & y & 1 & 0 \\ \bar{y} & \bar{x}\cup\bar{z} & 0 & 1 \end{bmatrix}$$

The original function f is not repeated in position 21 which is left blank until its final value is inserted at the end of the process.

A further example of greater complexity will now be given in order to demonstrate the power of the method which is not adequately revealed in the trivial cases so far considered. The example selected is given as a problem by Keister, Ritchie, and Washburn,[7] and its solution by a matrix method was demonstrated by Hohn and Schissler.[5]

Operating conditions for four leads with respect to the state of five relays are specified in accordance with table 9.9.2.

Relays operated	*Leads earthed*
v, w	1
v, x	2
w, x	1, 2
v, y	3
w, y	1, 3
x, y	2, 3
v, z	1, 2, 3
w, z	4
x, z	1, 4
y, z	2, 4
none	none

TABLE 9.9.2

It is assumed that no other input conditions will occur. Full use is made of "don't care" conditions in the simplification of the switching functions expressing the conditions under which the various leads are earthed. The resulting table is table 9.9.3.

Lead	*Switching function*
1	$(w\cap\bar{z})\cup(\bar{w}\cap\bar{y}\cap z)$
2	$(x\cap\bar{z})\cup(\bar{w}\cap\bar{x}\cap z)$
3	$(v\cap z)\cup(y\cap\bar{z})$
4	$\bar{v}\cap z$

TABLE 9.9.3

A matrix in which the fifth row and column represent connections to earth can now be written down:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & (w\cap\bar{z})\cup(\bar{w}\cap\bar{y}\cap z) \\ 0 & 1 & 0 & 0 & (x\cap\bar{z})\cup(\bar{w}\cap\bar{x}\cap z) \\ 0 & 0 & 1 & 0 & (v\cap z)\cup(y\cap\bar{z}) \\ 0 & 0 & 0 & 1 & \bar{v}\cap z \\ f_1 & f_2 & f_3 & f_4 & 1 \end{bmatrix}$$

Since z is a factor of a term in each component of the fifth column, a sixth node can be inserted to remove such terms. This, however, involves the temporary insertion of redundant terms in place of some of the zero components of the matrix. It is only necessary to show these terms above the diagonal as in the matrix which follows:

$$\begin{bmatrix} 1 & [\bar{w}\cap\bar{x}\cap\bar{y}] & [v\cap\bar{w}\cap\bar{y}] & [\bar{v}\cap\bar{w}\cap\bar{y}] & (w\cap\bar{z})\cup(\bar{w}\cap\bar{y}\cap z) & \bar{w}\cap\bar{y} \\ 0 & 1 & [v\cap\bar{w}\cap\bar{x}] & [\bar{v}\cap\bar{w}\cap\bar{x}] & (x\cap\bar{z})\cup(\bar{x}\cap\bar{w}\cap z) & \bar{w}\cap\bar{x} \\ 0 & 0 & 1 & 0 & (v\cap z)\cup(y\cap\bar{z}) & v \\ 0 & 0 & 0 & 1 & \bar{v}\cap z & \bar{v} \\ f_1 & f_2 & f_3 & f_4 & 1 & z \\ \bar{w}\cap\bar{y} & \bar{w}\cap\bar{x} & v & \bar{v} & z & 1 \end{bmatrix}$$

The redundant component 12 arises because the insertion of node 6 gives an additional path from node 1 to node 2 via node 6 when

$$(\bar{w}\cap\bar{y})\cap(\bar{w}\cap\bar{x}) = (\bar{w}\cap\bar{x}\cap\bar{y}) = 1$$

The remaining redundant components arise similarly. It is necessary here to check that the redundant components do not permit the earthing of a lead for conditions other than those prescribed. In this case, only the redundant component 23 gives an undesired path, hence the term $(v\cap z)$ in column 5 is not removed when node 6 is inserted. A further node, node 7, can now be inserted to remove terms in column 5 containing $\bar{z}$. After again checking the redundant terms, the following connection matrix is obtained:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \bar{w}\cap\bar{y} & w \\ 0 & 1 & 0 & 0 & 0 & \bar{w}\cap\bar{x} & x \\ 0 & 0 & 1 & 0 & v\cap z & 0 & y \\ 0 & 0 & 0 & 1 & 0 & \bar{v} & 0 \\ 0 & 0 & v\cap z & 0 & 1 & z & \bar{z} \\ \bar{w}\cap\bar{y} & \bar{w}\cap\bar{x} & 0 & \bar{v} & z & 1 & 0 \\ w & x & y & 0 & \bar{z} & 0 & 1 \end{bmatrix}$$

This matrix represents the network shown in figure 9.9.3.

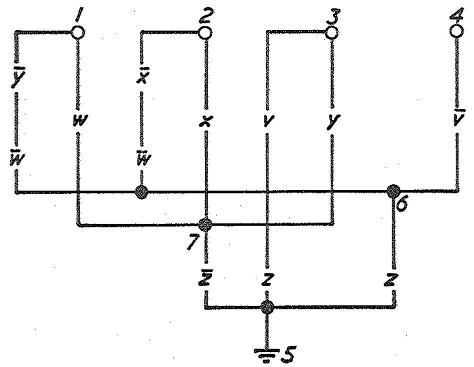

FIG. 9.9.3

It might appear at first sight that the two z-contacts next to earth could be combined, and likewise the two $\bar{w}$-contacts following node 6. Such combinations are not possible, however, since in the former instance node 2 would be earthed through node 3 when relays v and y are operated, and in the latter instance node 2 would be earthed through node 1 when relays v and w are operated. It is this type of error that the tests with redundant components are specifically designed to prevent.

In the examples considered so far bi-directional devices only have been utilized. Both analysis and synthesis techniques apply equally well to cases where uni-directional devices are involved, the only difference being that the switching matrices are not symmetrical.

Suppose that the requirement for a two-terminal circuit is given by the functions

$$f_{12} = (\bar{x} \cap y) \cup (y \cap z)$$
$$f_{21} = (x \cap z) \cup (y \cap z)$$

Since $f_{12} \neq f_{21}$, the output matrix F will not be symmetrical, and hence at least one uni-directional device will be required. It is, of course, possible to write down immediately a circuit satisfying the prescribed conditions using two uni-directional devices. However, the matrix approach to the problem yields initially

$$F = \begin{bmatrix} 1 & (\bar{x} \cap y) \cup (y \cap z) \\ (x \cap z) \cup (y \cap z) & 1 \end{bmatrix}$$

The first step is to insert a non-terminal node 3 to separate the y and z occurring in both output terms. This gives a connection matrix

$$\begin{bmatrix} 1 & \bar{x} \cap y & y \\ x \cap z & 1 & z \\ y & z & 1 \end{bmatrix}$$

A second non-terminal node, node 4, can now be inserted such that a path exists from node 1 to node 2 via nodes 3 and 4, thus allowing the deletion of component 12. Since components 13 and 31 are already y,

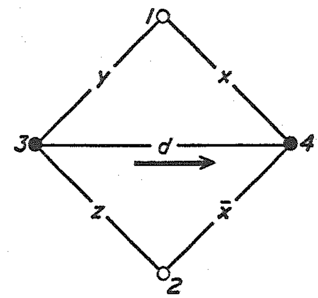

FIG. 9.9.4

component 34 can be made unity and component 42 can be made $\bar{x}$. However $(\bar{x} \cap y)$ is not a permitted path in the reverse direction, hence component 43 must be zero. The required switching conditions are now met by making components 14 and 41 take the value x, the primitive connection matrix being

$$\begin{bmatrix} 1 & 0 & y & x \\ 0 & 1 & z & \bar{x} \\ y & z & 1 & 1 \\ x & \bar{x} & 0 & 1 \end{bmatrix}$$

The required connection between nodes 3 and 4 can be obtained by the insertion of a simple uni-directional device. This can be considered for switching purposes to take the value 1 for forward transmission, and to take the value 0 for backward transmission. The network finally obtained is shown in figure 9.9.4.

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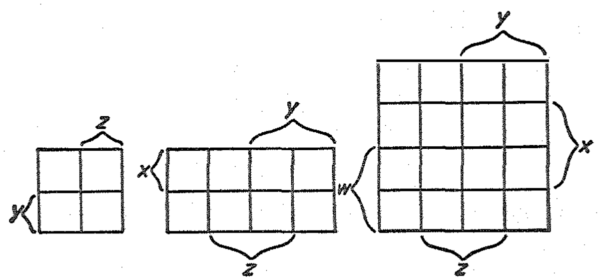
EXERCISES

1. Perform the simple matrix operations indicated, assuming in each case that the matrices represent entries on the Karnaugh maps, as appropriate, shown below:



(*a*) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

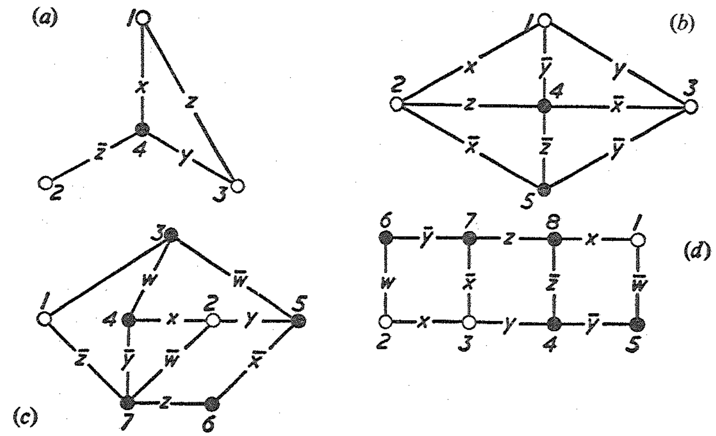
(*b*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(*d*) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cup \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$

(*e*) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cap \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \right)$

2. Write down the primitive connection matrix and the output matrix for each of the networks (*a*) to (*d*):



3. Form the matrices C_{-u}, as indicated, from the primitive connection matrices (*a*) to (*d*):

(*a*) C_{-5} from $\begin{bmatrix} 1 & 0 & x & \bar{y} & z \\ 0 & 1 & y & \bar{z} & 0 \\ x & y & 1 & 0 & \bar{x} \\ \bar{y} & \bar{z} & 0 & 1 & 0 \\ z & 0 & \bar{x} & 0 & 1 \end{bmatrix}$

(*b*) $C_{-4,\,3}$ from $\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & y & \bar{x} \\ 0 & y & 1 & z \\ x & \bar{x} & z & 1 \end{bmatrix}$

(*c*) $C_{-5,1}$ from $\begin{bmatrix} 1 & 0 & x & v & w & \bar{z} \\ 0 & 1 & 0 & 0 & \bar{v} & \bar{x} \\ x & 0 & 1 & \bar{w} & 0 & 0 \\ v & 0 & \bar{w} & 1 & x & z \\ w & \bar{v} & 0 & x & 1 & y \\ \bar{z} & \bar{x} & 0 & z & y & 1 \end{bmatrix}$

(*d*) $C_{-3,2}$ from $\begin{bmatrix} 1 & 0 & 0 & 0 & x & \bar{y} \\ 0 & 1 & \bar{x} & y & 0 & 0 \\ 0 & \bar{x} & 1 & z & \bar{z} & 0 \\ 0 & y & z & 1 & 0 & 0 \\ x & 0 & \bar{z} & 0 & 1 & x \\ \bar{y} & 0 & 0 & 0 & x & 1 \end{bmatrix}$

4. Obtain a primitive connection matrix by inserting the appropriate number of nodes into each of the connection matrices (*a*) to (*c*):

(*a*) $\begin{bmatrix} 1 & x\cup y & z \\ x\cup y & 1 & \bar{x}\cap\bar{y} \\ z & \bar{x}\cap\bar{y} & 1 \end{bmatrix}$

(*b*) $\begin{bmatrix} 1 & x & y \\ x & 1 & [\bar{x}\cap(\bar{y}\cup z)]\cup(\bar{y}\cap z) \\ y & [\bar{x}\cap(\bar{y}\cup z)]\cup(\bar{y}\cap z) & 1 \end{bmatrix}$

(*c*) The symmetric matrix, C, with components:

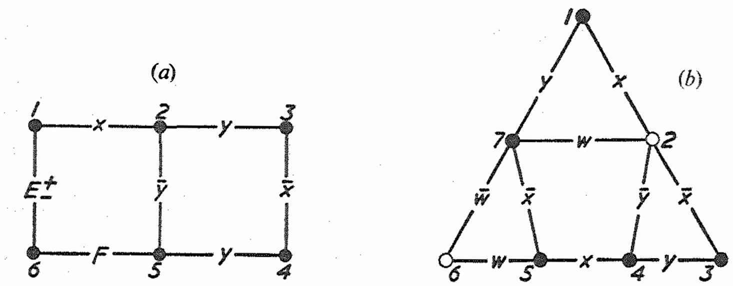
$$\begin{aligned} c_{11} &= 1 \\ c_{12} &= w\cap x \\ c_{13} &= y \\ c_{14} &= 0 \\ c_{15} &= 0 \\ c_{22} &= 1 \\ c_{23} &= \bar{w}\cap x\cap\bar{y} \\ c_{24} &= \bar{w}\cap x \\ c_{25} &= \bar{z}\cup(\bar{w}\cap x\cap y) \\ c_{33} &= 1 \\ c_{34} &= (\bar{w}\cap\bar{x})\cup(x\cap\bar{y}) \\ c_{35} &= 0 \\ c_{44} &= 1 \\ c_{45} &= (x\cap y)\cup(\bar{y}\cap z) \\ c_{55} &= 1 \end{aligned}$$

5. Expand and simplify the following determinants:

(*a*) $\det\begin{bmatrix} 1 & x & z \\ y & \bar{x} & 0 \\ z & 0 & 1 \end{bmatrix}$ (*b*) $\det\begin{bmatrix} x & y & z \\ 1 & \bar{x} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*c*) $\det\begin{bmatrix} 1 & x & y & 0 \\ y & \bar{x} & 1 & 0 \\ z & 0 & 0 & \bar{z} \\ 0 & \bar{y} & 0 & 1 \end{bmatrix}$ (*d*) $\det\begin{bmatrix} 1 & x & y & z \\ x & 1 & \bar{x} & 0 \\ y & \bar{x} & 1 & 0 \\ 0 & \bar{y} & 0 & \bar{z} \end{bmatrix}$

6. Determine the order relations defining all possible switching functions z, which may be substituted for the component $\bar{y}$ in the networks (*a*) and (*b*) without altering the output:

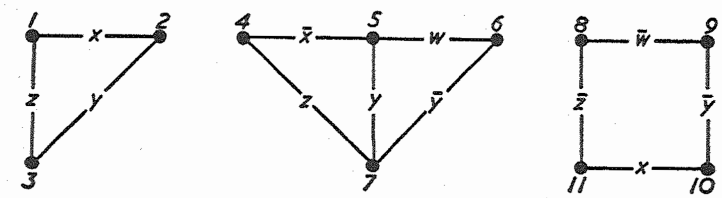


7. Write down the connection matrix for linking up the three networks shown in each of the ways indicated:

(*a*) 1–5, 2–4, 7–8, 6–9

(*b*) 1–5–9, 2–11, 3–6, 4–10

(*c*) 1–8, 3–7–9



8. Write down the output matrix and derive the primitive connection matrix which will satisfy the stated conditions in each of the following:

(*a*)

y	z	f_{12}	f_{13}	f_{14}	f_{23}	f_{24}	f_{34}
0	0	0	1	0	0	1	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	1	0	0	0	1	1	1

(*b*) $f=[\bar{y}\cap(w\cup\bar{x})]\cup[x\cap y\cap(\bar{w}\cup\bar{z})]$

(*c*) $f=[v\cap w\cap(x\cup z)]\cup\{x\cap[(v\cap z)\cup(w\cap\bar{z})]\}$

(*d*)

Relays operated (*of* 4)	*Leads earthed* (*of* 3)
w, x	1, 2
w, y	3
w, z	2
x, y	1, 3
x, z	1, 2, 3
y, z	1
none	none

No other inputs possible.

(*e*)

Relays operated (*of* 5)	*Leads earthed* (*of* 4)
v, w, x	1, 3
v, w, y	2
v, w, z	2, 4
v, x, y	1, 3, 4
v, x, z	1, 2
v, y, z	1, 4
w, x, y	2, 3
w, x, z	3
w, y, z	4
x, y, z	2, 3, 4
none	none

No other inputs possible.

(*f*) $f_{12} = \bar{x} \cup [\bar{z} \cap (x \cup \bar{y})]$
$f_{21} = \bar{y} \cup z$

(*g*) $f_{12} = \bar{w} \cup (x \cap y)$
$f_{13} = x$
$f_{23} = \bar{w} \cap x$
$f_{21} = \bar{w} \cup (\bar{x} \cap y)$
$f_{31} = 1$
$f_{32} = \bar{w} \cup \bar{x} \cup y$

10

Matrices with Boolean Components—II

10.1 Introductory

In section 9.2 three ways were described in which matrices with Boolean components can arise. This chapter will be devoted to matrices of the third type, namely those which arise in problems involving the solution of sets of simultaneous Boolean equations, and which have been termed *Boolean matrices* in order to distinguish them from the switching matrices already discussed at length in the preceding chapter. Boolean matrices of this type are particularly suited to the analysis and synthesis of digital systems employing delay elements, and such application will be described after consideration of the special algebraic properties of the matrices.

10.2 Basic Considerations

It has been seen that a Karnaugh map can give rise to a trivial type of matrix where the elements of the matrix take on the value 1 or 0 accordingly as there is or is not an entry in the map in the corresponding position. If the map is reshaped so that all the cells are in one horizontal line, the equivalent matrix will be a vector or row matrix of order 1×2^n. Several such maps placed one below another would therefore give rise to a matrix of order $p \times q$, where p is the number of functions plotted and $q = 2^n$ as before. Each cell of a Karnaugh map represents a minimal polynomial, hence the matrix components produced in this way will indicate the presence or absence of the various minimal polynomials in each function. Figure 10.2.1 shows two 1×2^2 maps with a switching function plotted upon each.

It will be seen that the decimal representations of the minimal polynomials are also indicated, and the two switching functions are clearly

$$f_1 = (\bar{y} \cap z) \cup (y \cap \bar{z})$$

and

$$f_2 = (\bar{y} \cap \bar{z}) \cup (y \cap \bar{z}) = \bar{z}$$

If y and z are now rewritten as z_2 and z_1 respectively, a familiar kind of matrix equation ensues, namely

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This may be written

$$Bz = f$$

where B is the matrix of coefficients and z and f are vectors. This represents the simultaneous Boolean equations

$$(0 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (0 \cap z_1 \cap z_2) = f_1$$
$$(1 \cap \bar{z}_1 \cap \bar{z}_2) \cup (0 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (0 \cap z_1 \cap z_2) = f_2$$

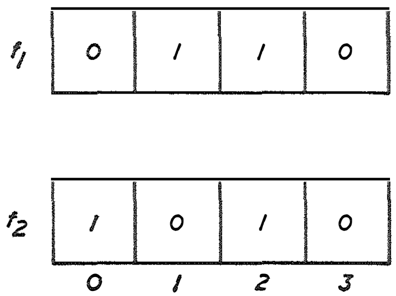


FIG. 10.2.1

A special type of vector can now be defined having decimal and not Boolean components. This is obtained from a Boolean matrix by regarding its columns as binary numbers, least significant digit at the top, and writing the decimal equivalent below each column. Thus, from the matrix

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

the vector

$$\mathbf{B} = [2 \quad 1 \quad 3 \quad 0]$$

is obtained.

The vector $\mathbf{A}$ is defined as having components $\mathbf{a}_s$ where

$$\mathbf{a}_s = s-1$$

and $$s = 1, 2, 3, \ldots, 2^n$$

Thus for $n = 2$

$$\mathbf{A} = [0 \quad 1 \quad 2 \quad 3]$$

and for $n = 3$

$$\mathbf{A} = [0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7]$$

Clearly, these can be obtained from matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

respectively. Clearly, a matrix A is of order $n \times 2^n$ and the binary digits 0 and 1 appear alternatively in row 1, in pairs in row 2, and so on up to groups of 2^{n-1} in row n.

It is sometimes useful to group the rows of a matrix A in pairs so that the columns produce sets of binary numbers and the vector $\mathbf{A}$ then becomes a matrix having more than one row. For example, when $n = 4$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If the columns are now taken to represent pairs of binary numbers

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{bmatrix}$$

where the suffix 2 indicates that the components within the columns of A have been taken two at a time. The matrix A for any value of n can be regarded as the identity matrix in the algebra of Boolean matrices in n-space.

Boolean matrices have certain properties in common with switching matrices and also with the matrices of ordinary algebra. Thus the join of two Boolean matrices corresponds to the join of two switching matrices; hence, if

$$D = B \cup C$$

then $$d_{rs} = b_{rs} \cup c_{rs}$$

and the indempotent, commutative, associative, and distributive laws will hold for Boolean matrix union. Further, the complement of a Boolean matrix B with components b_{rs} is the matrix $\bar{B}$ with components $\bar{b}_{rs}$, and the scalar product of a matrix B with some scalar c is a matrix with components $c \cap b_{rs}$.

Consider now any Boolean vector or column matrix z. Each of the 2^n possible z's will be equal to one and only one of the columns of the $n \times 2^n$ A-matrix. For example, if

$$z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

then the corresponding A-matrix will be

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is seen that only column 6 of A corresponds to the vector z. Such a column will be designated a_p.

It is now possible to define a special operation of Boolean matrix multiplication. Given the matrix product

$$BC = D$$

the columns of D are obtained one by one by comparing the columns of C with the appropriate A-matrix columns, i.e. locating respective a_p's, the appropriate column of D in each case being the pth column of B. For example, consider the product BC,

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

for which the appropriate A-matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The first column of C is the same as the second column of A, hence the first column of the product D is the same as the second column of B. The second column of C corresponds to column 1 of A, hence the

second column of D is the same as column 1 of B. The process is continued to yield the final product

$$D = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

The whole process may be carried out with even greater ease by using the decimal vector equivalents. Thus

$$\mathrm{BC} = [1 \quad 3 \quad 2 \quad 3] \quad [1 \quad 0 \quad 3 \quad 2]$$

yields the product

$$[3 \quad 1 \quad 3 \quad 2]$$

where each component is simply the $(\mathbf{c}_s + 1)$th component of B. As in the case of ordinary matrices, Boolean matrix multiplication is not commutative, neither does it obey the idempotent law. For the matrices B and C above, the product CB can be seen to be

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

which is not the same as the matrix D obtained previously.

A Boolean matrix is said to be *singular* when two or more of its columns are identical. A singular matrix has a defect of order d, where d is the number of columns of the identity matrix A not present in it. Thus the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

which has two pairs of identical columns, is singular and has a defect of order 2, since the fourth and seventh columns of the corresponding A-matrix are not present. As with ordinary matrices, the inverse of a Boolean matrix can be defined only in cases where the matrix is non-singular. The process of inversion may be carried out simply by a reversal of the process of multiplication. Since the A-matrix is an identity matrix, the inverse of a matrix B can be obtained from the equation

$$\mathbf{BB}^{-1} = \mathbf{A}$$

where the components of $\mathbf{B}^{-1}$ are given by subtracting 1 from the column number of $\mathbf{B}$ in which the number $(s-1)$ occurs for each $\mathbf{b}_s^{-1}$. Thus the inverse of the decimal vector

$$[1 \quad 2 \quad 0 \quad 3]$$

is the vector

$$[2 \quad 0 \quad 1 \quad 3]$$

The first component is seen to be $3-1$ since 0 occurs in the third column of $\mathbf{B}$. Clearly

$$\begin{aligned} & [1 \quad 2 \quad 0 \quad 3] \quad [2 \quad 0 \quad 1 \quad 3] \\ = & [2 \quad 0 \quad 1 \quad 3] \quad [1 \quad 2 \quad 0 \quad 3] \\ = & [0 \quad 1 \quad 2 \quad 3] \end{aligned}$$

Written in full, the above becomes

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Consider now the problem of finding the components of the vector f which satisfies the matrix equation

$$Bz = f$$

the components of B and z being stated explicitly. These are obtained by first locating column a_p of the appropriate A-matrix, f now being identical to the pth column of B. For example, given the matrix equation, $Bz = f$, as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

and referring to the appropriate A-matrix, namely

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

column 6 of A corresponds to the vector

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and hence the vector f is the same as column 6 of B, namely

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A special case of matrix equation is

$$Bz = z$$

and any vector z which satisfies this equality is termed a *characteristic vector* of B. For example, both

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are characteristic vectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

since

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The processes, which have been described above in this section, will be expressed formally in the section which follows.

10.3 Formalization of Basic Processes

The vector $\mathbf{A}$ with components $0, 1, 2, \ldots, 2^n$ may be formally defined as that vector having components $\mathbf{a}_s$ where

$$\mathbf{a}_s = \sum_{r=1}^{n} 2^{r-1} a_{rs}; \; s = 1, 2, \ldots, 2^n$$

where a_{rs} are the components of the A-matrix already described. It can easily be seen that this is effectively the formula for binary-to-decimal conversion. In a similar manner, a vector $\mathbf{B}$ can be defined as that vector having components $\mathbb{b}_s$ where

$$\mathbb{b}_s = \sum_{r=1}^{n} 2^{r-1} b_{rs}; \; s = 1, 2, \ldots, 2^n$$

b_{rs} being the components of any matrix B.

A set of simultaneous Boolean equations can be expressed in the generalized form

$$F_r(z_1, z_2, \ldots, z_n) = f_r; \; r = 1, 2, \ldots, n$$

The components of the matrix of coefficients, B say, are given by

$$b_{rs} = F_r(a_{1s}, a_{2s}, \ldots, a_{ns}); \; r = 1, 2, \ldots, n \quad s = 1, 2, \ldots, 2^n$$

This is merely a formalized algebraic representation of the process of obtaining minimal polynomials. For example, consider the equations

$$z_1 \cup \bar{z}_2 = f_1$$
$$(z_1 \cap \bar{z}_2) \cup (\bar{z}_1 \cap z_2) = f_2$$

The component b_{13} of the matrix of coefficients, B, is given by

$$\begin{aligned} b_{13} &= F_1(a_{13}, a_{23}) \\ &= F_1(0, 1) \\ &= 0 \cup 0 \\ &= 0 \end{aligned}$$

Similarly, the component b_{22} of B is given by

$$\begin{aligned} b_{22} &= F_2(a_{12}, a_{22}) \\ &= F_2(1, 0) \\ &= (1 \cap 1) \cup (0 \cap 0) \\ &= 1 \end{aligned}$$

These results may be checked by expressing the equations in their canonical form and comparing the coefficients of the minimal polynomials with those obtained by the formalized algebraic process.

As an example of the reverse process, consider the Boolean matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This represents the set of equations in canonical form

$$(1 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (0 \cap z_1 \cap z_2) = f_1$$
$$(0 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (1 \cap z_1 \cap z_2) = f_2$$

These are obtained by use of the expression, quoted by Campeau[1] but here stated in a slightly different form

$$f_r = \bigcup_{s=1}^{2^n} b_{rs} \cap \bigcap_{t=1}^{n} [(a_{ts} \cap z_t) \cup (\bar{a}_{ts} \cap \bar{z}_t)]; \; r = 1, 2, \ldots, n$$

the components a_{ts} being the components of the A-matrix.

This expression can be regarded as a definition of the multiplication of a vector z with components $z_r (r = 1, 2, \ldots, n)$ by a matrix B with components $b_{rs} (r = 1, 2, \ldots, n; \; s = 1, 2, \ldots, 2^n)$. A simpler expression can however be found if the "inner product" of two Boolean vectors, y and z (say) with components y_r and z_r respectively, is defined as

$$(y, z) = \bigcap_{r=1}^{n} [(y_r \cap z_r) \cup (\bar{y}_r \cap \bar{z}_r)]$$

The expression for f_r can now be written as

$$f = \bigcup_{s=1}^{2^n} b_s \cap (z, a_s)$$

where f and z are vectors with components f_r and $z_r (r = 1, 2, \ldots, n)$, and b_s and a_s are the columns of the B-matrix and the identity matrix A respectively.

It is, of course, always possible and by far the simpler method to write down the canonical equations directly from the matrix equation. Nevertheless, the example which follows is quoted in order to illustrate the use of the symbolical expression for f_r. Reverting to the B-matrix of the equation given shortly above, namely

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

the first equation is given by

$$\begin{aligned} f_1 = & \bigcup_{s=1}^{4} b_{1s} \cap \bigcap_{t=1}^{2} [(a_{ts} \cap z_t) \cup (\bar{a}_{ts} \cap \bar{z}_t)] \\ = & \quad \{b_{11} \cap [(a_{11} \cap z_1) \cup (\bar{a}_{11} \cap \bar{z}_1)] \cap [(a_{21} \cap z_2) \cup (\bar{a}_{21} \cap \bar{z}_2)]\} \\ & \cup \{b_{12} \cap [(a_{12} \cap z_1) \cup (\bar{a}_{12} \cap \bar{z}_1)] \cap [(a_{22} \cap z_2) \cup (\bar{a}_{22} \cap \bar{z}_2)]\} \\ & \cup \{b_{13} \cap [(a_{13} \cap z_1) \cup (\bar{a}_{13} \cap \bar{z}_1)] \cap [(a_{23} \cap z_2) \cup (\bar{a}_{23} \cap \bar{z}_2)]\} \\ & \cup \{b_{14} \cap [(a_{14} \cap z_1) \cup (\bar{a}_{14} \cap \bar{z}_1)] \cap [(a_{24} \cap z_2) \cup (\bar{a}_{24} \cap \bar{z}_2)]\} \end{aligned}$$

Substituting in the appropriate values from the matrices A and B yields

$$\begin{aligned} f_1 = \; & \{1\cap[(0\cap z_1)\cup(1\cap\bar{z}_1)]\cap[(0\cap z_2)\cup(1\cap\bar{z}_2)]\} \\ & \cup\{1\cap[(1\cap z_1)\cup(0\cap\bar{z}_1)]\cap[(0\cap z_2)\cup(1\cap\bar{z}_2)]\} \\ & \cup\{1\cap[(0\cap z_1)\cup(1\cap\bar{z}_1)]\cap[(1\cap z_2)\cup(0\cap\bar{z}_2)]\} \\ & \cup\{0\cap[(1\cap z_1)\cup(0\cap\bar{z}_1)]\cap[(1\cap z_2)\cup(0\cap\bar{z}_2)]\} \end{aligned}$$

whence

$$f_1 = (1\cap\bar{z}_1\cap\bar{z}_2)\cup(1\cap z_1\cap\bar{z}_2)\cup(1\cap\bar{z}_1\cap z_2)\cup(0\cap z_1\cap z_2)$$

This is the first of the two Boolean equations already quoted.

A simple practical method for obtaining the product of two matrices was described in section 10.2. Consider now the matrix equation

$$B(Cy) = z$$

expressed in the vector form

$$z = \bigcup_{s=1}^{2^n} b_s\cap(Cy, a_s)$$

Since only one column of A will correspond to any particular vector y, namely column a_p, the equation can be rewritten as

$$z = \bigcup_{s=1}^{2^n} b_s\cap(c_p, a_s)$$

Now $(a_k, y) = 1$ only for $k = p$, hence the equality is not invalidated if the meet of the right-hand side with (a_p, y) is taken together with the join with

$$\bigcup_{k=1}^{2^n}\left[\bigcup_{s=1}^{2^n} b_s\cap(a_s, c_k)\right]\cap(a_k, y);\; k\neq p$$

Simplification yields

$$z = \bigcup_{k=1}^{2^n}\left[\bigcup_{s=1}^{2^n} b_s\cap(c_k, a_s)\right]\cap(a_k, y)$$

If a Boolean matrix D is now defined having columns

$$d_k = \bigcup_{s=1}^{2^n} b_s\cap(c_k, a_s);\; k = 1, 2, \ldots, 2^n$$

then the equation f●r z becomes

$$z = \bigcup_{k=1}^{2^n} d_k\cap(a_k, y)$$

which is equivalent to

$$Dy = z$$

where

$$B(Cy) = (BC)y = Dy$$

A formal algebraic definition of Boolean matrix multiplication has thus been made. The development of this formalized process is due to Campeau,[1] who defines the expression for d_{rk} in terms of the matrices A, B, C in a form equivalent to

$$d_{rk} = \bigcup_{s=1}^{2^n} b_{rs}\cap\bigcap_{t=1}^{n}[(c_{tk}\cap a_{ts})\cup(\bar{c}_{tk}\cap\bar{a}_{ts})]$$

$$r = 1, 2, \ldots, n \qquad k = 1, 2, \ldots, 2^n$$

In the example quoted in section 10.2

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The 11 component of the matrix D, where $BC = D$, is therefore given by

$$\begin{aligned} d_{11} &= \bigcup_{s=1}^{4} b_{1s}\cap\bigcap_{t=1}^{2}[(c_{t1}\cap a_{ts})\cup(\bar{c}_{t1}\cap\bar{a}_{ts})] \\ &= [1\cap(0\cup 0)\cap(0\cup 1)]\cup[1\cap(1\cup 0)\cap(0\cup 1)] \\ &\quad\cup[0\cap(0\cup 0)\cap(0\cup 0)]\cup[1\cap(1\cup 0)\cap(0\cup 0)] \\ &= 0\cup 1\cup 0\cup 0 \\ &= 1 \end{aligned}$$

If the terms in the square brackets are extracted from the general expression for d_{rk}, namely

$$(c_{tk}\cap a_{ts})\cup(\bar{c}_{tk}\cap\bar{a}_{ts})$$

it can be seen that this is in fact

$$c_{tk}\equiv a_{ts}$$

The general expression can therefore be written as

$$d_{rk} = \bigcup_{s=1}^{2^n} b_{rs} \cap \bigcap_{t=1}^{n} (c_{tk} \equiv a_{ts})$$

as elsewhere proposed by the author.[2] Using this form of the expression, the component d_{24}, for example, is given by

$$\begin{aligned} d_{24} &= \bigcup_{s=1}^{4} b_{2s} \cap \bigcap_{t=1}^{2} (c_{t4} \equiv a_{ts}) \\ &= (0 \cap 1 \cap 0) \cup (1 \cap 0 \cap 0) \cup (1 \cap 1 \cap 1) \cup (1 \cap 0 \cap 1) \\ &= 0 \cup 0 \cup 1 \cup 0 \\ &= 1 \end{aligned}$$

The determinant of a Boolean matrix is defined by

$$\det B = \bigcap_{s=1}^{2^n} \bigcup_{k=1}^{2^n} (b_k, a_s)$$

A singular matrix may now be defined as a matrix whose determinant is zero. For example, the determinant of the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

is given by

$$\begin{aligned} \det B &= \bigcap_{s=1}^{4} \bigcup_{k=1}^{4} (b_k, a_s) \\ &= 0 \cap 1 \cap 1 \cap 1 \\ &= 0 \end{aligned}$$

hence the matrix is singular. An expression for the inverse of a Boolean matrix can now be given, analogous to that for the inverse in ordinary matrix algebra

$$b_s^{-1} = \frac{1}{\det B} \cap \left[\bigcup_{k=1}^{2^n} a_k \cap (b_k, a_s) \right]; \ s = 1, 2, \ldots, 2^n$$

where

$$\tfrac{1}{1} = 1$$

and $\frac{1}{0}$ is not defined. For example, consider the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

for which the determinant det B can be found to be equal to unity. The columns of B^{-1} are given by

$$b_s^{-1} = 1 \cap \left[\bigcup_{k=1}^{4} a_k \cap (b_k, a_s) \right]$$

which, by a process analogous to the method of obtaining a Boolean matrix product one column at a time, gives

$$B^{-1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

10.4 Output Vectors of Switching Circuits

The output of any switching circuit taken across each pair of terminal nodes for a given set of input conditions can be obtained as a vector f. For example, consider the very simple network shown in figure 10.4.1. This yields the equations

$$\begin{aligned} z_1 \cup \bar{z}_2 &= f_{12} \\ z_1 \cup z_2 &= f_{13} \\ z_1 &= f_{23} \end{aligned}$$

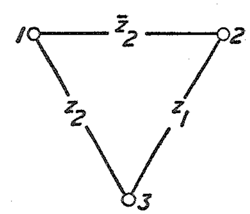


FIG. 10.4.1

Expressed in canonical form these equations become

$$\begin{aligned} (1 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (0 \cap \bar{z}_1 \cap z_2) \cup (1 \cap z_1 \cap z_2) &= f_{12} \\ (0 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (1 \cap \bar{z}_1 \cap z_2) \cup (1 \cap z_1 \cap z_2) &= f_{13} \\ (0 \cap \bar{z}_1 \cap \bar{z}_2) \cup (1 \cap z_1 \cap \bar{z}_2) \cup (0 \cap \bar{z}_1 \cap z_2) \cup (1 \cap z_1 \cap z_2) &= f_{23} \end{aligned}$$

These yield the Boolean matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} = \begin{bmatrix} f_{12} \\ f_{13} \\ f_{23} \end{bmatrix}$$

It should be noticed here that the vector z contains only two components, whereas the vector f has three. The appropriate A-matrix is therefore the 2×4 matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The vector z corresponds to the first column of the A-matrix if, for example, neither z_1 nor z_2 are operated, and the equation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} f_{12} \\ f_{13} \\ f_{23} \end{bmatrix}$$

therefore has the solution

$$f = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

i.e. there will be an output only across nodes 1 and 2 for the given condition of z_1 and z_2. That this is so can readily be seen from the figure.

Now let the input conditions be z_1 only operated. The matrix equation becomes

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} f_{12} \\ f_{13} \\ f_{23} \end{bmatrix}$$

for which the solution

$$f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is obtained. In this case there will be an output across every pair of nodes, which reference to figure 10.4.1 again shows to be the case. Completing the analysis yields

$$f = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for the input conditions z_2 only operated, and

$$f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

if both z_1 and z_2 are operated.

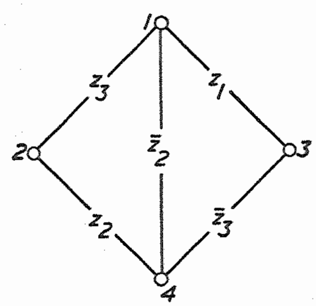


FIG. 10.4.2

As a second example, consider the network shown in figure 10.4.2 which has already been considered in sections 9.4 and 9.7.

$$\begin{aligned} (z_1 \cap z_2) \cup z_3 &= f_{12} \\ z_1 \cup (\bar{z}_2 \cap \bar{z}_3) &= f_{13} \\ z_1 \cup \bar{z}_2 \cup z_3 &= f_{14} \\ (z_1 \cap z_3) \cup (z_2 \cap \bar{z}_3) &= f_{23} \\ z_2 \cup z_3 &= f_{24} \\ z_1 \cup z_2 \cup \bar{z}_3 &= f_{34} \end{aligned}$$

These yield the matrix equation

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \\ z_2 \\ \\ z_3 \end{bmatrix} = \begin{bmatrix} f_{12} \\ f_{13} \\ f_{14} \\ f_{23} \\ f_{24} \\ f_{34} \end{bmatrix}$$

Since the vector z contains three elements, the appropriate A-matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The output vector f can now be obtained for any given set of input conditions. Thus if z_2 only is operated

$$z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which is equivalent to the third column of the A-matrix; the output is

$$f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The operation is greatly facilitated by using the decimal vectors described in section 10.2. The matrix equation then becomes

$$[38 \quad 38 \quad 56 \quad 63 \quad 21 \quad 63 \quad 53 \quad 63] \quad \mathbf{z} = \mathbf{f}$$

and the appropriate vector A-matrix is now

$$[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7]$$

If z_2 only is operated, $\mathbf{z} = 2$, and the output is therefore the $(2+1)$th, i.e. the third term of the coefficient decimal vector. Thus $\mathbf{f}$ is 56, and the output conditions between all the pairs of terminal nodes are given by the binary equivalent of 56.

The problem under consideration may be stated in a way which is the reverse of the two cases examined above. Suppose, for example, that the output conditions are specified for the network of figure 10.4.2 by $\mathbf{f} = 21$, i.e. a signal is required across nodes 1 and 2, 1 and 4, and 2 and 4 only. The fifth component of the decimal vector describing the network is equal to 21, hence the necessary input conditions are given by $\mathbf{z} = 5-1 = 4$, i.e. z_3 only must be operated. It should be noted that output conditions other than those equivalent to columns of the coefficient matrix, i.e. terms of the decimal vector, are impossible. For instance, if output conditions denoted by $\mathbf{f} = 12$ are demanded of the same network, the evaluation of the necessary input conditions is halted immediately by the fact that $\mathbf{f}$ cannot be matched with any component of

$$[38 \quad 38 \quad 56 \quad 63 \quad 21 \quad 63 \quad 53 \quad 63]$$

On the other hand, more than one set of input conditions may give rise to a prescribed output. Suppose that a signal is required across nodes 1 and 3, and 1 and 4, and therefore across nodes 3 and 4 also. This yields $\mathbf{f} = 38$, which corresponds to both the first and second components of the decimal vector. Hence, $\mathbf{z}$ may be either 0 or 1, and the conditions are met if no input elements are operated or if z_1 only is operated. Clearly, output conditions giving rise to $\mathbf{f} = 63$ could be met in three different ways, since 63 appears in three places in the decimal vector.

10.5 Digital Systems with Delay Elements

The type of matrix with Boolean components described in this chapter is particularly suited to the analysis and synthesis of digital systems employing delay elements. Thus, if $z_1, z_2, \ldots, z_n$ represent the states of n single delay elements of a system at time instant t, then

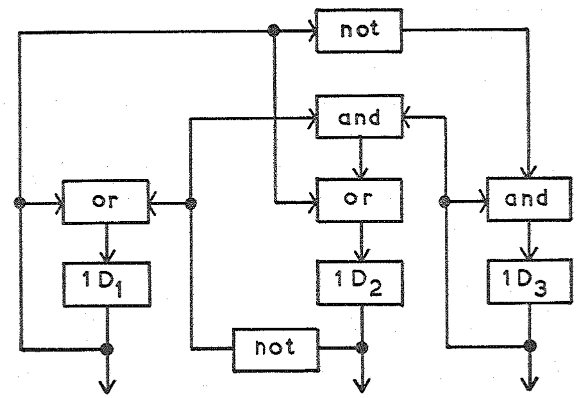


FIG. 10.5.1

$f_1, f_2, \ldots, f_n$ can represent the states of the same elements at time instant $t+1$, and the operation of the system from one time instant to the next is represented by a Boolean matrix B.

Consider the system represented in figure 10.5.1. The corresponding set of Boolean equations is

$$z_1 \cup \bar{z}_2 = f_1$$
$$z_1 \cup (\bar{z}_2 \cap z_3) = f_2$$
$$\bar{z}_1 \cap z_3 = f_3$$

The matrix equation is therefore

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

The reaction of the system to any given set of input conditions z_1, z_2, z_3 may now be investigated. For example, let the initial input be

$$z_1 = z_2 = z_3 = 0$$

The equation now becomes

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \end{bmatrix}$$

Since the vector z corresponds to the first column of the appropriate A-matrix, the solution of the equation is obtained from the first column of the coefficient matrix, giving

$$f_1 = 1; f_2 - f_3 = 0$$

The vector f thus represents the state of the output after one clock pulse.

It is now possible to put this output as a new set of input conditions in order to obtain the output state after two clock pulses. The equation now becomes

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

The solution of this equation is seen to be

$$f_1 = f_2 = 1; f_3 = 0$$

These values may then be fed into the equation in their turn, and the process continued until a complete output cycle for input conditions, $z_1 = z_2 = z_3 = 0$, is obtained. This is shown in table 10.5.1.

f_1	f_2	f_3	*Time*
0	0	0	t (input)
1	0	0	$t+1$
1	1	0	$t+2$
1	1	0	$>t+2$

TABLE 10.5.1

It can thus be seen that after two clock pulses the system will "lock" with a permanent output

$$f_1 = f_2 = 1; f_3 = 0$$

Again, if the initial conditions are given by

$$z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

the cycle obtained is that given in table 10.5.2.

f_1	f_2	f_3	*Time*
0	1	0	t (input)
0	0	0	$t+1$
1	0	0	$t+2$
1	1	0	$\geqq t+3$

TABLE 10.5.2

A complete investigation of the output cycles for each possible set of input conditions in fact shows that the system will lock on 1, 1, 0 after not more than three clock pulses whatever the input.

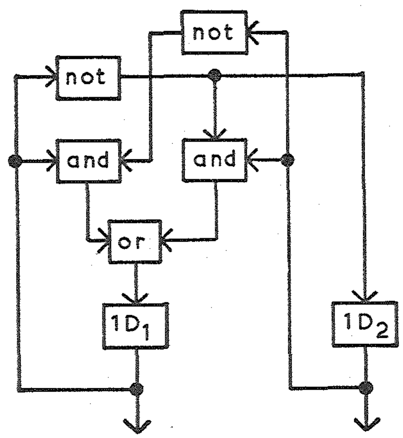


FIG. 10.5.2

A second digital system will now be considered. This is illustrated in figure 10.5.2. The set of Boolean equations for this system is

$$(z_1 \cap \bar{z}_2) \cup (\bar{z}_1 \cap z_2) = f_1$$
$$\bar{z}_1 = f_2$$

Hence the matrix equation is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

If the initial conditions, $z_1 = z_2 = 0$, are postulated, then the cycle of table 10.5.3 results.

f_1	f_2	*Time*
0	0	$t+3k$
0	1	$t+1+3k$
1	1	$t+2+3k$

TABLE 10.5.3

If initially $z_1 = 1$, $z_2 = 0$ it is seen that the system will lock immediately on this output.

The analysis becomes even more simple if decimal notation is adopted. Thus in the system just considered

$$\mathrm{B} = [2 \quad 1 \quad 3 \quad 0]$$

The first set of initial conditions is equivalent to

$$\mathbf{z} = 0$$

and since this is the same as the first component of

$$\mathbf{A} = [0 \quad 1 \quad 2 \quad 3]$$

the output at $t+1$ is the first component of $\mathbf{B}$, i.e. 2. The output cycle is now obtained as

$$0,\ 2,\ 3,\ 0,\ 2,\ 3,\ 0,\ \ldots$$

which is the decimal equivalent of that obtained previously.

It should be noticed that the matrix for the system of figure 10.5.1 is singular, but the matrix for the system of figure 10.5.2 is non-singular. Since a singular matrix must have at least two columns identical, it is clear that at least one out of the 2^n theoretical output vectors is not present as a column of the system matrix, and hence can never represent the output whatever the input conditions. It should also be noticed that a locking state occurred in both systems when a particular column of B was identical to the corresponding column of A. Clearly, this must always be the case, and such a state must be equivalent to a characteristic vector as defined in section 10.2, since

$$Bf = f$$

If the output is to be sampled not at every clock pulse but only at alternate clock pulses commencing with $t+2$, then clearly this is equivalent to setting up a system represented by

$$B^2 z = f$$

For example, referring again to the system illustrated in figure 10.5.2, if

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

then

$$B^2 = BB = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

or alternatively

$$\mathbf{B}^2 = [3 \quad 1 \quad 0 \quad 2]$$

If the input conditions represented by $\mathbf{z} = 0$ are now applied, the output cycle

$$0,\ 3,\ 2,\ 0,\ 3,\ 2,\ 0,\ \ldots$$

is obtained. This represents sampling at every other clock pulse as can be seen by reference to the output cycle obtained previously for the same input.

In the example just considered, since a three-stage cycle was obtained

$$\begin{aligned} B &= B^4 = B^7 = \ldots = B^{1+3k};\ k = 0, 1, 2, \ldots \\ B^2 &= B^5 = B^8 = \ldots = B^{2+3k} \\ B^3 &= B^6 = B^9 = \ldots = B^{3+3k} \end{aligned}$$

and these matrices will each have

$$\mathbf{b}_2 = \mathbf{a}_2 = 1$$

In general, if it is required to know the state of a digital system after k clock pulses, then this is given by the relationship

$$B^k z = f$$

For example, consider a system for which

$$\mathrm{B} = [2 \quad 1 \quad 3 \quad 2 \quad 0 \quad 7 \quad 4 \quad 5]$$

Evaluation of successive powers of **B** yields

$$\begin{aligned}
\mathbf{B}^2 &= [3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 5\ \ 0\ \ 7] \\
\mathbf{B}^3 &= [2\ \ 1\ \ 3\ \ 2\ \ 3\ \ 7\ \ 2\ \ 5] \\
\mathbf{B}^4 &= [3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 5\ \ 3\ \ 7] \\
\mathbf{B}^5 &= [2\ \ 1\ \ 3\ \ 2\ \ 3\ \ 7\ \ 2\ \ 5] \\
\mathbf{B}^6 &= [3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 5\ \ 3\ \ 7]
\end{aligned}$$

and in general

$$\begin{aligned}
\mathbf{B}^{3+2k} &= \mathbf{B}^3;\ k = 1, 2, \ldots \\
\mathbf{B}^{4+2k} &= \mathbf{B}^4
\end{aligned}$$

Thus, the output after 5 clock pulses for an input $\mathbf{z} = 7$ is $\mathbf{f} = 5$. Much more than this can be ascertained by examination of the various $\mathbf{B}^k$. Clearly, the system will lock immediately for an input $\mathbf{z} = 1$. Further, whatever other input is applied the system will quickly settle down to repeating some pair of outputs. A complete analysis of the system yields the following cycles:

$$\begin{aligned}
&0, 2, 3, 2, 3, 2, \ldots \\
&1, 1, 1, 1, 1, 1, \ldots \\
&2, 3, 2, 3, 2, 3, \ldots \\
&3, 2, 3, 2, 3, 2, \ldots \\
&4, 0, 2, 3, 2, 3, \ldots \\
&5, 7, 5, 7, 5, 7, \ldots \\
&6, 4, 0, 2, 3, 2, \ldots \\
&7, 5, 7, 5, 7, 5, \ldots
\end{aligned}$$

These may be represented as shown in figure 10.5.3.

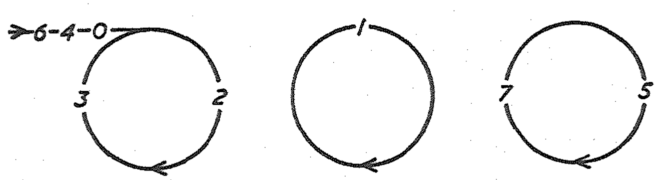


FIG. 10.5.3

The inverse of the matrix B of a system represents that system for which the output cycle is reversed. For example, the system with

$$\mathbf{B} = [1\ \ 2\ \ 0\ \ 3]$$

responds to an input $\mathbf{z} = 0$ with a cycle

$$0, 1, 2, 0, 1, 2, 0, \ldots$$

The inverse of **B** has been found earlier to be

$$\mathbf{B}^{-1} = [2\ \ 0\ \ 1\ \ 3]$$

and this responds to $\mathbf{z} = 0$ with the cycle

$$0, 2, 1, 0, 2, 1, \ldots$$

In each case an input $\mathbf{z} = 3$ leads to immediate locking.

10.6 Synthesis using Decimal Vectors

The simplest synthesis case is that of designing the logic for a system with a prescribed cycle. Suppose that a digital system employing delay elements is required having an output cycle,

$$1, 3, 6, 2, 5, 7, 1, \ldots$$

together with a locking state 4. The latter requirement immediately indicates that $\mathbf{b}_5 = 4$. It is also apparent that the B-matrix will be of order 3×8, and that there must be three outputs.

The next stage is to write down the appropriate **A**-vector, with the output state next due according to the prescribed cycle underneath each $\mathbf{a}_s$. In this case, the process yields

$$\begin{matrix}
[0 & 1 & 2 & 3 & 4 & 5 & 6 & 7] \\
[? & 3 & 5 & 6 & 4 & 7 & 2 & 1]
\end{matrix}$$

The lower vector is the required **B**, which expands to

$$B = \begin{bmatrix} ? & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ ? & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ ? & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The first column represents the output one clock pulse after an input $\mathbf{z} = 0$. The system can be made to lock on $\mathbf{f} = 0$ for zero input by making $\mathbf{b}_1 = 0$. Alternatively, it could be made to enter the cycle at any desired point or to lock on $\mathbf{f} = 4$. If the input $\mathbf{z} = 0$ is not possible, it is desirable to leave the first column indeterminate and to use the minimal polynomials represented by the indeterminate components as "don't care" conditions, thus aiding the minimization process which now follows. Adopting this last principle in the example being considered yields, after minimization, the set of equations:

$$(\bar{z}_1 \cap \bar{z}_3) \cup [z_1 \cap (\bar{z}_2 \cup z_3)] = f_1$$
$$[z_1 \cap (\bar{z}_2 \cup \bar{z}_3)] \cup (\bar{z}_1 \cap z_2 \cap z_3) = f_2$$
$$(z_2 \cap \bar{z}_3) \cup (\bar{z}_2 \cap z_3) = f_3$$

which, in turn, represents the system shown in figure 10.6.1.

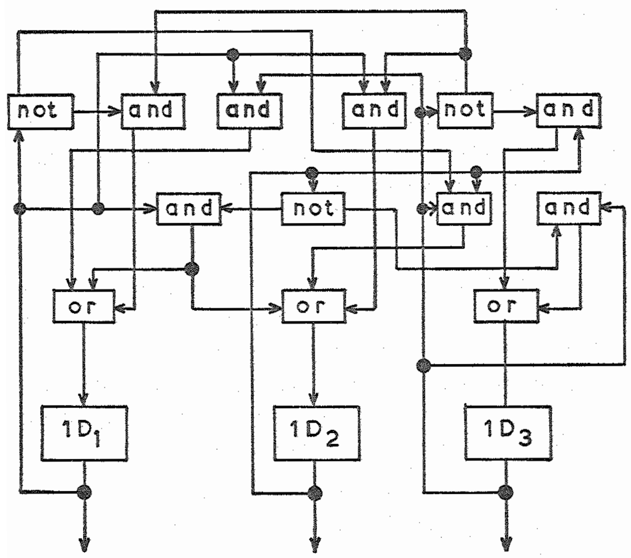


FIG. 10.6.1

As a second example, consider the design of a simple parallel adder which replaces any two numbers up to three by their sum. The **A**-matrix

$$\mathbf{A} = [0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9 \;\; 10 \;\; 11 \;\; 12 \;\; 13 \;\; 14 \;\; 15]$$

can be written, as has been demonstrated in section 10.2, in the form

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Each term $\mathbf{b}_s$ is now obtained as the sum of the components in column s of $\mathbf{A}_2$. Thus

$$\mathbf{B} = [0 \;\; 1 \;\; 2 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 3 \;\; 4 \;\; 5 \;\; 6]$$

which expands to

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

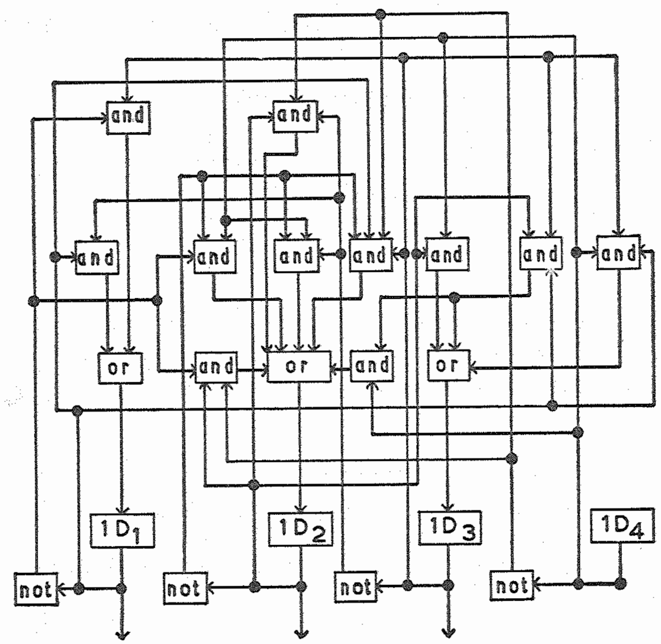


FIG. 10.6.2

This yields, after minimization, the set of equations

$$(z_1 \cap \bar{z}_3) \cup (\bar{z}_1 \cap z_3) = f_1$$
$$(\bar{z}_1 \cap z_2 \cap \bar{z}_4) \cup (\bar{z}_1 \cap \bar{z}_2 \cap z_4) \cup (z_2 \cap \bar{z}_3 \cap \bar{z}_4) \cup (\bar{z}_2 \cap \bar{z}_3 \cap z_4)$$
$$\cup (z_1 \cap z_2 \cap z_3 \cap z_4) \cup (z_1 \cap \bar{z}_2 \cap z_3 \cap \bar{z}_4) = f_2$$
$$(z_2 \cap z_4) \cup (z_1 \cap z_2 \cap z_3) \cup (z_1 \cap z_3 \cap z_4) = f_3$$
$$0 = f_4$$

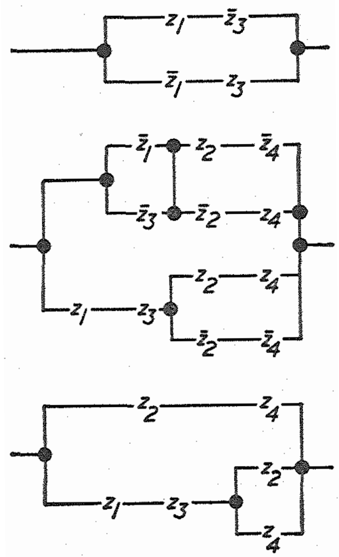


FIG. 10.6.3

The digital system represented by these equations and incorporating delay elements is shown in figure 10.6.2, and a switching network producing the same output is shown in figure 10.6.3.

REFERENCES

1. J. O. CAMPEAU, "The Synthesis and Analysis of Digital Systems by Boolean Matrices", *Trans. IRE*, Vol. EC–6, 1957, pp. 231–41.
2. H. G. FLEGG, "The Manipulation and Minimisation of Boolean Switching Functions", College of Aeronautics Thesis, 1959.

EXERCISES

1. Write down the full matrix equation and also the equivalent in decimal vector form for each of the following sets of Boolean equations:

(*a*)
$$\begin{aligned} \bar{z}_3\cap(\bar{z}_1\cup z_2) &= f_1\\ (\bar{z}_2\cap z_3)\cup(\bar{z}_1\cap z_2\cap\bar{z}_3) &= f_2\\ (\bar{z}_1\cap\bar{z}_2)\cup\bar{z}_3 &= f_3 \end{aligned}$$

(*b*)
$$\begin{aligned} z_1\cup(\bar{z}_2\cap z_3) &= f_1\\ \bar{z}_2\cap(\bar{z}_1\cup\bar{z}_4) &= f_2\\ (z_2\cap\bar{z}_4)\cap[(z_1\cap\bar{z}_3)\cup(\bar{z}_1\cap z_3)] &= f_3\\ (\bar{z}_2\cap\bar{z}_3\cap\bar{z}_4)\cup(z_1\cap z_2\cap z_3\cap z_4) &= f_4 \end{aligned}$$

(*c*)
$$\begin{aligned} (z_1\cap z_5)\cap[(z_2\cap\bar{z}_3\cap z_4)\cup(\bar{z}_2\cap z_3\cap\bar{z}_4)] &= f_1\\ \bar{z}_1\cup(z_2\cap\bar{z}_3)\cup(\bar{z}_2\cap z_4\cap z_5) &= f_2\\ (\bar{z}_2\cap\bar{z}_3)\cap(z_4\cup\bar{z}_5) &= f_3\\ \bar{z}_1\cap z_2\cap z_3\cap z_5 &= f_4\\ (\bar{z}_2\cap z_3)\cup(z_1\cap z_2\cap\bar{z}_3\cap z_4\cap z_5) &= f_5 \end{aligned}$$

2. Determine the Boolean matrix D in each of the following:

(*a*) $\begin{bmatrix} 0&0&1&0&1&0&0&1\\ 1&0&1&1&1&0&0&0\\ 0&1&1&0&0&0&1&0 \end{bmatrix} \cup \begin{bmatrix} 1&0&0&0&1&0&1&1\\ 1&0&0&1&1&0&1&0\\ 1&1&1&1&0&0&1&1 \end{bmatrix} = D$

(*b*) $\begin{bmatrix} 1&0&0&0\\ 0&1&0&1 \end{bmatrix} \begin{bmatrix} 0&1&0&0\\ 1&0&0&1 \end{bmatrix} = D$

(*c*) $[0\ \ 0\ \ 3\ \ 1]\ \ [1\ \ 2\ \ 3\ \ 0] = \mathbf{D}$

(*d*) $[7\ \ 1\ \ 6\ \ 3\ \ 2\ \ 0\ \ 4\ \ 5]\ \ [6\ \ 7\ \ 0\ \ 1\ \ 4\ \ 2\ \ 5\ \ 3] = \mathbf{D}$

(*e*) $[3\ \ 1\ \ 0\ \ 2]^{-1} = \mathbf{D}$

(*f*) $[5\ \ 4\ \ 1\ \ 3\ \ 0\ \ 6\ \ 7\ \ 2]^{-1} = \mathbf{D}$

(*g*) $[0\ \ 3\ \ 2\ \ 2\ \ 5\ \ 7\ \ 5\ \ 2]^{3} = \mathbf{D}$

(*h*) $\begin{bmatrix} 1&0&1&0&0&0&0&0&1&1&1&1&0&0&1&1\\ 0&1&1&1&1&0&1&0&1&0&0&0&0&0&1&1\\ 1&0&0&0&1&1&0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&1&1&0&0&1 \end{bmatrix}^{5} = D$

3. Carry out a complete analysis of the systems represented by the following matrices or decimal vectors, in each case writing the output cycles in the form used in figure 10.5.3:

(*a*) $\begin{bmatrix} 1&0&0&1&1&1&0&0\\ 0&1&0&0&1&1&1&1\\ 0&0&1&1&0&0&1&0 \end{bmatrix}$ sampled at every clock pulse.

(*b*) $\begin{bmatrix} 0&0&0&1&1&0&0&1\\ 1&0&1&0&1&1&0&1\\ 0&0&1&1&0&0&1&1 \end{bmatrix}$ sampled at every other clock pulse $t+2k$.

(*c*) $[3\ \ 1\ \ 2\ \ 5\ \ 4\ \ 0\ \ 0\ \ 3\ \ 8\ \ 9\ \ 11\ \ 13\ \ 15\ \ 15\ \ 8\ \ 8]$ sampled at every clock pulse.

(*d*) $[7\ \ 3\ \ 5\ \ 6\ \ 3\ \ 2\ \ 7\ \ 0]$ sampled at clock pulses $t+3+4k$.

(*e*) $\begin{bmatrix} 0&1&1&0&0&0&0&1&1&1&1&0&0&0&1&0&1&1&0&1&1&1&1&1&1&0&0&0&1&0&1&1\\ 1&1&0&1&1&1&0&1&0&1&0&1&1&0&1&1&0&0&0&1&0&1&0&1&1&0&0&1&1&0&1&0\\ 0&1&1&1&0&1&1&1&1&0&0&0&0&1&1&1&1&0&1&1&0&0&1&1&0&1&1&0&1&1&0&0\\ 1&1&1&0&1&1&1&0&0&1&0&1&1&0&0&1&1&0&1&1&0&0&0&0&0&1&1&1&1&0&1&0\\ 1&1&1&0&1&0&0&1&0&1&1&0&1&0&1&1&0&1&1&0&0&0&1&1&1&0&1&1&0&1&1&1 \end{bmatrix}$

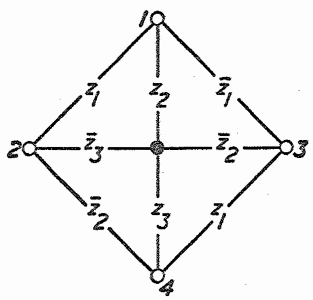
sampled at every clock pulse, and also sampled at every third clock pulse $t+2+3k$.

4. For the networks (*a*) to (*c*), shown below, write down the sets of Boolean equations in canonical forms which define the signals across each pair of terminal nodes, and determine the output vectors, f, for the inputs specified:

(*a*)

Inputs: (i) $z_1 = z_2 = z_3 = 0$

(ii) $z = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

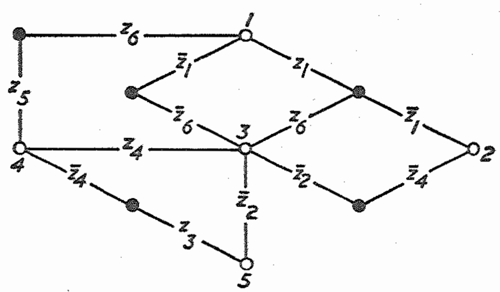


(*b*)

Inputs: (i) $z_r = 0$; $r = 1, 2, \ldots, 6$

(ii) $z_1 = z_5 = 1$, $z_r = 0$; $r = 3, 4, 5, 6$

(iii) $z = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

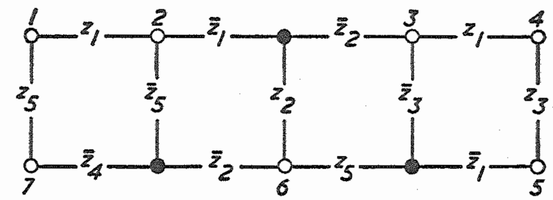


(*c*)

Inputs: (i) $z_r = 1$; $r = 1, 2, \ldots, 5$

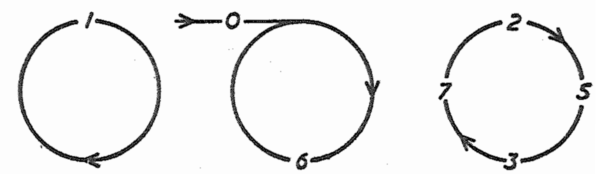
(ii) $z = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(iii) $z = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

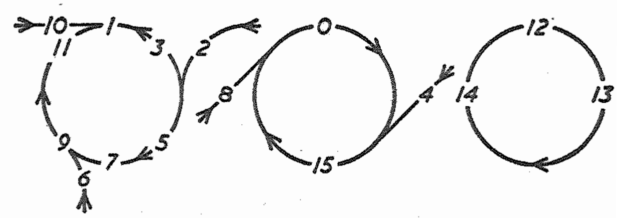


5. Derive the appropriate *B*-matrix and also the set of minimized Boolean equations for the digital systems having outputs as indicated in (*a*) to (*d*) below:

(*a*)



(*b*)



(*c*) 1–5–2–7–0–3–4–6–1– ... ,
sampled every third clock pulse.

(*d*) Input: $z = 0, 1, \ldots, 7$
Output after one clock pulse: z^2.

6. Sketch the system

(*a*) of question 5(*a*), and

(*b*) of question 5(*d*),
adopting the form of diagram as utilized in figure 10.6.2 and basing the diagrams on the derived minimized Boolean equations.

APPENDIX A

VARIOUS CONVENTIONS FOR COMMON LOGICAL OPERATIONS

Operation	*Other Names*	*Symbols*
Denial	Complementation	$\bar{z}$
	Negation	z'
		z^*
		$\sim z$
		Nz
Conjunction	Intersection	$y \cap z$
	Meet	$y \wedge z$
	And	$y.z$
		yz
		Kyz
Alternation	Inclusive Disjunction	$y \cup z$
	Union	$y \vee z$
	Join	$y+z$
	Or	Ayz
Implication	Conditional	$y \leqq z$
	Less than or equal to	$y \rightarrow z$
		$y \supset z$
		Cyz
Inclusion	Converse Implication	$y \geqq z$
	Greater than or equal to	$y \subset z$
		Lyz
Equivalence	Biconditional	$y \equiv z$
		$y \leftrightarrow z$
		Eyz
Non-conjunction	Sheffer Stroke	$y \mid z$
		$y \bar{\cap} z$
		$y \bar{\wedge} z$
		Dyz

Non-alternation	Non-disjunction	$y \overline{\cup} z$
		$y \overline{\nabla} z$
		Syz
Non-implication	Greater than	$y > z$
		$y \nleqq z$
		$y \not\supset z$
		Hyz
Non-inclusion	Converse Non-implication	$y < z$
		$y \ngeqq z$
		$y \not\subset z$
		Tyz
Non-equivalence	Sum Module Two	$y \not\equiv z$
	Symmetric Difference	$y \oplus z$
	Exclusive Disjunction	$y \triangle z$
	Exclusive Or	Ryz

APPENDIX B

MATRICES

A matrix is a rectangular array of numbers which is subject to certain rules of combination with other such arrays. The general $m \times n$ matrix has m rows and n columns and, written with literal terms, appears as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Such a matrix may be designated by A or, alternatively, by $[a_{ij}]$ where i takes the successive values 1, 2, 3, . . . , m, and j takes the successive values 1, 2, 3, . . . , n. When $m = n$ the matrix is said to be square and of order n.

Certain square matrices are of special interest, in particular those where all terms are zero except terms on the principal diagonal; the principal diagonal being defined as a_{ij}, $i = j$. Such matrices are termed diagonal matrices, an important diagonal matrix being the unit matrix, U,

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}$$

Addition of matrices of equal order $m \times n$ is carried out by adding corresponding terms of the two matrices. Thus

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Multiplication of a matrix by a scalar quantity is effected by multiplying each term of the matrix by the scalar. Thus

$$k[a_{ij}] = [ka_{ij}]$$

2. (*a*) $S(0)_{0,3}f(x, \bar{y}, z) = 1$
$S(0)_{1,2}f(x, \bar{y}, z) = 0$
$S(1)_{1,2}f(x, \bar{y}, z) = 0$

(*b*) $S(1)_{0,1,4}f(v, w, x, \bar{y}, \bar{z}) = 1$
$S(1)_{2,3,5}f(v, w. x, \bar{y}, \bar{z}) = 0$
$S(0)_{0,2,3}f(v, w, x, \bar{y}, \bar{z}) = 0$

(*c*) $S(1)_{0,1,3}f(\bar{w}, \bar{x}, y, \bar{z}) = 0$
$S(1)_{2,4}f(\bar{w}, \bar{x}, y, \bar{z}) = 1$
$S(0)_{0,2}f(\bar{w}, \bar{x}, y, \bar{z}) = 1$

3. (*a*) $S(1)_{0,3}f(x, \bar{y}, z) = 0$
$S(0)_{0,3}f(x, \bar{y}, z) = 0$
$S(0)_{0,3}f(\bar{x}, y, \bar{z}) = 0$
$S(0)_{1,2}f(x, \bar{y}, z) = 1$
$S(1)_{0,3}f(\bar{x}, y, \bar{z}) = 0$
$S(1)_{1,2}f(x, \bar{y}, z) = 1$
$S(1)_{1,2,}f(\bar{x}\ y, \bar{z}) = 1$
$S(0)_{1,2}f(\bar{x}, y, \bar{z}) = 1$

(*b*) $S(0)_{1,4,5}f(v, w, x, \bar{y}, \bar{z}) = 0$
$S(1)_{0,1,4}f(v, w, x, \bar{y}, \bar{z}) = 0$
$S(1)_{1,4,5}f(\bar{v}, \bar{w}, \bar{x}, y, z) = 0$
$S(1)_{2,3,5}f(v, w, x, \bar{y}, \bar{z}) = 1$
$S(0)_{0,1,4}f(\bar{v}, \bar{w}, \bar{x}, y, z) = 0$
$S(0)_{0,2,3}f(v, w, x, \bar{y}, \bar{z}) = 1$
$S(0)_{2,3,5}f(\bar{v}, \bar{w}, \bar{x}, y, z) = 1$
$S(1)_{0,2,3}f(\bar{v}, \bar{w}, \bar{x}, y, z) = 1$

(*c*) $S(0)_{1,3,4}f(\bar{w}, \bar{x}, y, \bar{z}) = 1$
$S(1)_{0,1,3}f(\bar{w}, \bar{x}, y, \bar{z}) = 1$
$S(1)_{1,3,4}f(w, x, \bar{y}, z) = 1$
$S(1)_{2,4}f(\bar{w}, \bar{x}, y, \bar{z}) = 0$
$S(0)_{0,1,3}f(w, x, \bar{y}, z) = 1$
$S(0)_{0,2}f(\bar{w}, \bar{x}, y, \bar{z}) = 0$
$S(0)_{2,4}f(w, x, \bar{y}, z) = 0$
$S(1)_{0,2}f(w, x, \bar{y}, z) = 0$

4. (*a*) Intersection: $S(0)_3 \bigcup (w \cap x \cap y \cap z) = 1$
Union: $S(0)_{0,1,3,4} \bigcup (w \cap x \cap y \cap z) = 1$

(*b*) Intersection: $S(0)_{4,5} \bigcup (v \cap \bar{w} \cap \bar{x} \cap y \cap z) = 0$
Union: $S(0)_{0,1,4,5} \bigcup (v \cap \bar{w} \cap \bar{x} \cap y \cap z) = 0$

(*c*) Intersection: 0
Union: $S(0)_{1,2,3,4,5,6} \bigcup (\bar{u} \cap v \cap w \cap x \cap y \cap z) = 1$

(*d*) Intersection: $S(0)_2 \bigcup (w \cap x \cap y \cap z) = 1$
Union: I

5. (*a*) Symmetric, (*b*) Not symmetric, (*c*) Symmetric.

6. (*a*) Symmetric, (*b*) Symmetric, (*c*) Symmetric, (*d*) Not symmetric.

7. (*a*)
$$\begin{aligned}&(v \cap w \cap \bar{x} \cap \bar{y} \cap \bar{z}) \cup (\bar{v} \cap \bar{w} \cap \bar{x} \cap \bar{y} \cap \bar{z}) \cup (\bar{v} \cap w \cap x \cap \bar{y} \cap \bar{z})\\ &\cup (\bar{v} \cap w \cap \bar{x} \cap y \cap \bar{z}) \cup (\bar{v} \cap w \cap \bar{x} \cap \bar{y} \cap z)\\ &\cup (v \cap \bar{w} \cap x \cap \bar{y} \cap \bar{z}) \cup (v \cap \bar{w} \cap \bar{x} \cap y \cap \bar{z})\\ &\cup (v \cap \bar{w} \cap \bar{x} \cap \bar{y} \cap z) \cup (v \cap w \cap x \cap y \cap \bar{z})\\ &\cup (v \cap w \cap x \cap \bar{y} \cap z) \cup (v \cap w \cap \bar{x} \cap y \cap z)\end{aligned}$$

(*b*)
$$\begin{aligned}&(v \cap w \cap x \cap \bar{y}) \cup (\bar{v} \cap w \cap x \cap z) \cup (w \cap \bar{x} \cap y \cap z) \cup (v \cap \bar{w} \cap y \cap z)\\ &\cup (v \cap x \cap y \cap \bar{z}) \cup (v \cap w \cap \bar{x} \cap z) \cup (v \cap \bar{w} \cap x \cap z)\\ &\cup (v \cap w \cap y \cap \bar{z}) \cup (\bar{v} \cap x \cap y \cap z) \cup (\bar{v} \cap w \cap x \cap y)\end{aligned}$$

(*c*) $\bar{w} \cup x \cup \bar{y} \cup \bar{z}$

Chapter 9

1. (*a*) $\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\end{bmatrix}$ (*b*) $\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{bmatrix}$

(*c*) $\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\end{bmatrix}$ (*d*) $\begin{bmatrix}1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}$ (*e*) $\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{bmatrix}$

2. (*a*)
$$P = \begin{bmatrix}1 & 0 & z & x\\ 0 & 1 & 0 & \bar{z}\\ z & 0 & 1 & y\\ x & \bar{z} & y & 1\end{bmatrix} \qquad F = \begin{bmatrix}1 & x \cap \bar{z} & (x \cap y) \cup z\\ x \cap \bar{z} & 1 & y \cap \bar{z}\\ (x \cap y) \cup z & y \cap \bar{z} & 1\end{bmatrix}$$

(*b*)
$$P = \begin{bmatrix}1 & x & y & \bar{y} & 0\\ x & 1 & 0 & z & \bar{x}\\ y & 0 & 1 & \bar{x} & \bar{y}\\ \bar{y} & z & \bar{x} & 1 & \bar{z}\\ 0 & \bar{x} & \bar{y} & \bar{z} & 1\end{bmatrix} \qquad F = \begin{bmatrix}1 & \bar{x} \cup \bar{y} & \bar{x} \cup y \cup \bar{z}\\ \bar{x} \cup \bar{y} & 1 & \bar{x} \cup y \cup \bar{z}\\ \bar{x} \cup y \cup \bar{z} & \bar{x} \cup y \cup \bar{z} & 1\end{bmatrix}$$

(*c*)
$$P = \begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0 & \bar{z}\\ 0 & 1 & 0 & x & y & 0 & \bar{w}\\ 1 & 0 & 1 & w & \bar{w} & 0 & 0\\ 0 & x & w & 1 & 0 & 0 & \bar{y}\\ 0 & y & \bar{w} & 0 & 1 & \bar{x} & 0\\ 0 & 0 & 0 & 0 & \bar{x} & 1 & z\\ \bar{z} & \bar{w} & 0 & \bar{y} & 0 & z & 1\end{bmatrix}$$

$$F = \begin{bmatrix}1 & [\bar{w} \cap (\bar{x} \cup y \cup \bar{z})] \cup (w \cap x)\\ [\bar{w} \cap (\bar{x} \cup y \cup \bar{z})] \cup (w \cap x) & 1\end{bmatrix}$$

(*d*)
$$P = \begin{bmatrix}1 & 0 & 0 & 0 & \bar{w} & 0 & 0 & x\\ 0 & 1 & x & 0 & 0 & w & 0 & 0\\ 0 & x & 1 & y & 0 & 0 & \bar{x} & 0\\ 0 & 0 & y & 1 & \bar{y} & 0 & 0 & \bar{z}\\ \bar{w} & 0 & 0 & \bar{y} & 1 & 0 & 0 & 0\\ 0 & w & 0 & 0 & 0 & 1 & \bar{y} & 0\\ 0 & 0 & \bar{x} & 0 & 0 & \bar{y} & 1 & z\\ x & 0 & 0 & \bar{z} & 0 & 0 & z & 1\end{bmatrix}$$

$$F = \begin{bmatrix}1 & x \cap [(y \cap \bar{z}) \cup (w \cap \bar{y} \cap z)] & x \cap [(y \cap \bar{z}) \cup (w \cap \bar{y} \cap z)]\\ f_{21} & 1 & x \cup (w \cap \bar{y})\\ f_{31} & f_{32} & 1\end{bmatrix}$$

3. (*a*) $\begin{bmatrix}1 & 0 & x \cup z & \bar{y}\\ 0 & 1 & y & \bar{z}\\ x \cup z & y & 1 & 0\\ \bar{y} & \bar{z} & 0 & 1\end{bmatrix}$ (*b*) $\begin{bmatrix}1 & x \cap y \cap z\\ x \cap y \cap z & 1\end{bmatrix}$

(c) $$\begin{bmatrix} 1 & x\cap\bar{v}\cap w & \bar{v}\cap x & \bar{x}\cup\{\bar{v}\cap[y\cup(w\cap\bar{z})]\} \\ c_{21} & 1 & \bar{w}\cap x & x\cap[\bar{z}\cup(w\cap y)] \\ c_{31} & c_{32} & 1 & v\cup z\cup[x\cap(w\cup y)] \\ c_{41} & c_{42} & c_{43} & 1 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & x & \bar{y} \\ 0 & 1 & \bar{x}\cap y\cap\bar{z} & 0 \\ x & \bar{x}\cap y\cap\bar{z} & 1 & x \\ \bar{y} & 0 & x & 1 \end{bmatrix}$$

4. (a) $$\begin{bmatrix} 1 & x & z & y \\ x & 1 & 0 & \bar{x} \\ z & 0 & 1 & \bar{y} \\ y & \bar{x} & \bar{y} & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & x & y & 0 & 0 \\ x & 1 & 0 & \bar{y} & \bar{x} \\ y & 0 & 1 & \bar{x} & \bar{y} \\ 0 & \bar{y} & \bar{x} & 1 & z \\ 0 & \bar{x} & \bar{y} & z & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & y & 0 & 0 & w & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \bar{z} & x & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 & 0 & \bar{y} & \bar{w} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x & \bar{x} & \bar{y} \\ 0 & \bar{z} & 0 & 0 & 1 & 0 & y & 0 & z \\ w & x & 0 & 0 & 0 & 1 & \bar{w} & 0 & 0 \\ 0 & 0 & \bar{y} & x & y & \bar{w} & 1 & 0 & 0 \\ 0 & 0 & \bar{w} & \bar{x} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{y} & z & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. (a) $\bar{x}\cup y$, (b) y, (c) $[(x\cup y)\cap z]\cup(\bar{y}\cap\bar{z})$, (d) 1.

6. (a) $(\bar{x}\cap\bar{y})\leqq z\leqq(x\cup\bar{y})$, (b) $(w\cap x\cap\bar{y})\leqq z\leqq(\bar{w}\cup\bar{x}\cup\bar{y})$.

7. (a) $$\begin{bmatrix} 1 & 1 & z & w & y & 0 & 0 \\ 1 & 1 & y & 0 & z & 0 & 0 \\ z & y & 1 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & 1 & \bar{w}\cup\bar{y} & \bar{y} & 0 \\ y & z & 0 & \bar{w}\cup\bar{y} & 1 & 0 & \bar{z} \\ 0 & 0 & 0 & \bar{y} & 0 & 1 & x \\ 0 & 0 & 0 & 0 & \bar{z} & x & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & x & w\cup z & \bar{x}\cup\bar{y} & y & \bar{w} \\ x & 1 & y & x & 0 & \bar{z} \\ w\cup z & y & 1 & 0 & \bar{y} & 0 \\ \bar{x}\cup\bar{y} & x & 0 & 1 & z & 0 \\ y & 0 & \bar{y} & z & 1 & 0 \\ \bar{w} & \bar{z} & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & x & \bar{w}\cup z & 0 & 0 & 0 & 0 & \bar{z} \\ x & 1 & y & 0 & 0 & 0 & 0 & 0 \\ \bar{w}\cup z & y & 1 & z & y & \bar{y} & \bar{y} & 0 \\ 0 & 0 & z & 1 & \bar{x} & 0 & 0 & 0 \\ 0 & 0 & y & \bar{x} & 1 & w & 0 & 0 \\ 0 & 0 & \bar{y} & 0 & w & 1 & 0 & 0 \\ 0 & 0 & \bar{y} & 0 & 0 & 0 & 1 & x \\ \bar{z} & 0 & 0 & 0 & 0 & 0 & x & 1 \end{bmatrix}$$

8. (a) $$\begin{bmatrix} 1 & 0 & \bar{y} & 0 & 0 & y \\ 0 & 1 & z & 0 & 0 & \bar{y} \\ \bar{y} & z & 1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & z & \bar{z} \\ 0 & 0 & y & z & 1 & 0 \\ y & \bar{y} & 0 & \bar{z} & 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & w\cup\bar{x} & \bar{w}\cup\bar{z} & 0 \\ 0 & 1 & \bar{y} & 0 & x \\ w\cup\bar{x} & \bar{y} & 1 & 0 & 0 \\ \bar{w}\cup\bar{z} & 0 & 0 & 1 & y \\ 0 & x & 0 & y & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 0 & 0 & w & v & z \\ 0 & 1 & x\cup z & x & 0 & 0 & 0 \\ 0 & x\cup z & 1 & 0 & 0 & w & 0 \\ 0 & x & 0 & 1 & \bar{z} & 0 & v \\ w & 0 & 0 & \bar{z} & 1 & 0 & 0 \\ v & 0 & w & 0 & 0 & 1 & 0 \\ z & 0 & 0 & v & 0 & 0 & 1 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \bar{y} & 0 \\ 0 & 0 & 1 & 0 & x & 0 & y \\ x & 0 & 0 & 1 & z & x\cup z & \bar{z} \\ y & 0 & x & z & 1 & 0 & 0 \\ 0 & \bar{y} & 0 & x\cup z & 0 & 1 & 0 \\ 0 & 0 & y & \bar{z} & 0 & 0 & 1 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & \bar{w} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{y} & \bar{v} & w \\ 0 & 0 & 1 & 0 & 0 & 0 & w & 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \bar{w}\cup z & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & v & x & v & y & 0 & 0 & 0 \\ x & 0 & 0 & 0 & v & 1 & 0 & 0 & 0 & z & 0 & \bar{w} \\ 0 & 0 & w & 0 & x & 0 & 1 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & v & 0 & 0 & 1 & 0 & 0 & 0 & \bar{x} \\ 0 & 0 & 0 & \bar{w}\cup z & y & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \bar{y} & 0 & w & 0 & z & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \bar{v} & y & 0 & 0 & 0 & y & 0 & 0 & 0 & 1 & 0 \\ \bar{w} & w & 0 & 0 & 0 & \bar{w} & 0 & \bar{x} & 0 & 0 & 0 & 1 \end{bmatrix}$$

(f) $$\begin{bmatrix} 1 & 0 & z & \bar{y} & 1 \\ 0 & 1 & 1 & \bar{z} & \bar{x} \\ z & 0 & 1 & 0 & y \\ \bar{y} & \bar{z} & 0 & 1 & x \\ 0 & \bar{x} & y & x & 1 \end{bmatrix}$$

(g) $$\begin{bmatrix} 1 & 0 & 0 & \bar{w} & 0 & w & 0 & 0 & 0 \\ 0 & 1 & 0 & \bar{x} & y & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & 0 & \bar{x} & 0 & \bar{z} \\ \bar{w} & \bar{x} & x & 1 & \bar{y} & x & 0 & y & z \\ 0 & y & 0 & \bar{y} & 1 & 0 & 0 & 0 & 0 \\ w & \bar{w} & 0 & x & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{x} & 1 & 0 & 0 & 1 & \bar{y} & 0 \\ 1 & 0 & 0 & y & 0 & 0 & \bar{y} & 1 & 0 \\ 0 & 0 & \bar{z} & z & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 10

1. (*a*) $$\begin{bmatrix}1&0&1&1&0&0&0&0\\0&0&1&0&1&1&0&0\\1&1&1&1&1&0&0&0\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}$$

$$[5\quad 4\quad 7\quad 5\quad 6\quad 1\quad 0\quad 0]\quad \mathbf{z}=\mathbf{f}$$

(*b*) $$\begin{bmatrix}0&1&0&1&1&1&0&1&0&1&0&1&1&1&0&1\\1&1&0&0&1&1&0&0&1&0&0&0&1&0&0&0\\0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}=\begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix}$$

$$[10\quad 11\quad 0\quad 5\quad 3\quad 3\quad 4\quad 1\quad 2\quad 1\quad 0\quad 1\quad 3\quad 1\quad 0\quad 9]\quad \mathbf{z}=\mathbf{f}$$

(*c*)

$$\left[\begin{matrix}0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\1&0&1&1&1&0&1&0&1&0&1&1&1&0&1&0&1&0&1&1&1&0\\1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\end{matrix}\right.$$

$$\left.\begin{matrix}0&0&0&0&0&1&0&0&0&0\\1&0&1&1&1&1&1&1&1&0\\0&0&1&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&1&1&0&0\end{matrix}\right]\begin{bmatrix}z_1\\z_2\\z_3\\z_4\\z_5\end{bmatrix}=\begin{bmatrix}f_1\\f_2\\f_3\\f_4\\f_5\end{bmatrix}$$

$$[6\quad 4\quad 2\quad 2\quad 18\quad 16\quad 2\quad 0\quad 6\quad 4\quad 2\quad 2\quad 18\quad 16\quad 2\quad 0\quad 2\quad 0\quad 2\quad 2\quad 18$$
$$17\quad 10\quad 0\quad 6\quad 6\quad 2\quad 19\quad 18\quad 18\quad 10\quad 0]\quad \mathbf{z}=\mathbf{f}$$

2. (*a*) $$\begin{bmatrix}1&0&1&0&1&0&1&1\\1&0&1&1&1&0&1&0\\1&1&1&1&0&0&1&1\end{bmatrix}$$

(*b*) $$\begin{bmatrix}0&0&1&0\\0&1&0&0\end{bmatrix}$$

(*c*) $[0\quad 3\quad 1\quad 0]$

(*d*) $[4\quad 5\quad 7\quad 1\quad 2\quad 6\quad 0\quad 3]$

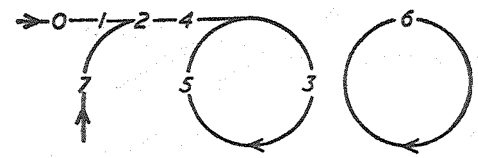
(*e*) $[2\quad 1\quad 3\quad 0]$

(*f*) $[4\quad 2\quad 7\quad 3\quad 1\quad 0\quad 5\quad 6]$

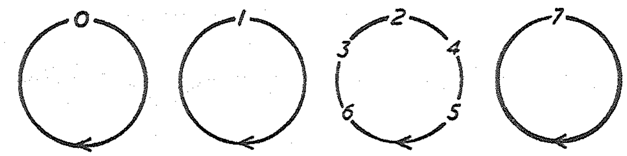
(*g*) $[0\quad 2\quad 2\quad 2\quad 2\quad 2\quad 2\quad 2]$

(*h*) $$\begin{bmatrix}0&0&1&0&0&1&0&1&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1\\0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\end{bmatrix}$$

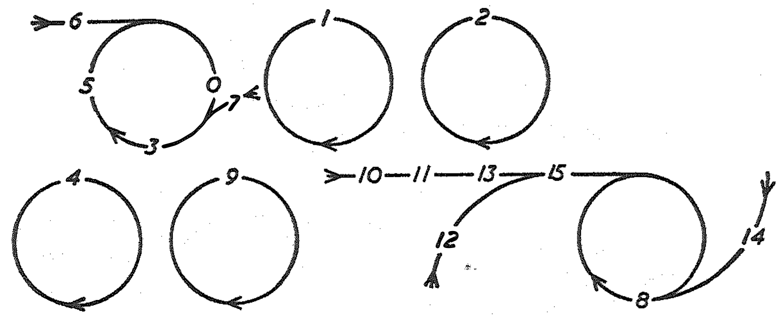
3. (*a*)



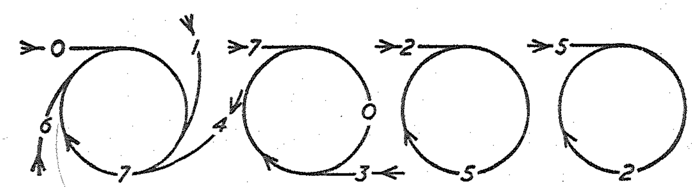
(*b*)



(*c*)

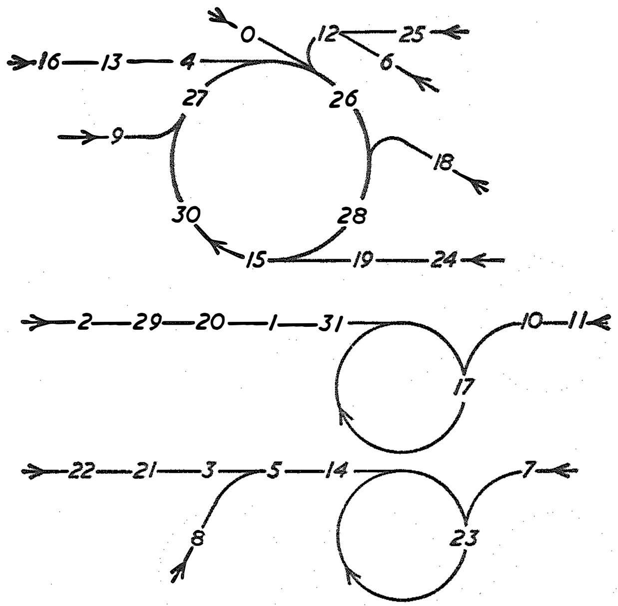


(*d*)

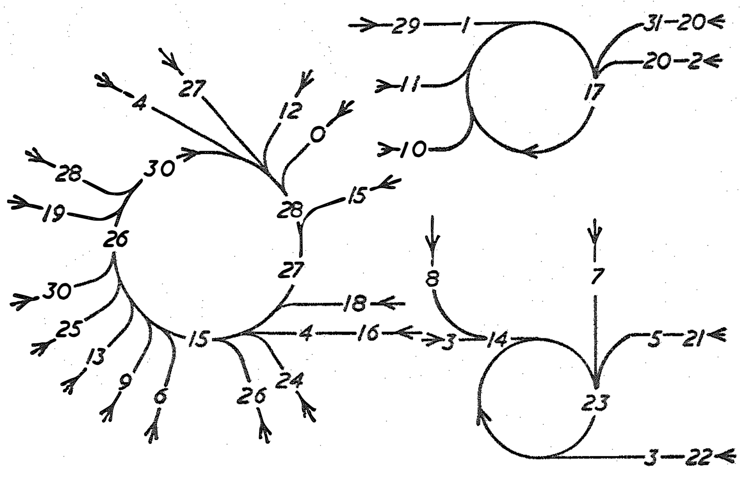


Note: In parts (*a*), (*b*) and (*c*), each number occurs once only in the solution and hence there is no ambiguity. In parts (*d*) and (*e*), however, some numbers occur more than once. When this is so, the output cycle for a given initial input is obtained by selecting the relevant number where it occurs at the beginning of an input line.

(e)



for sampling at every clock pulse



for sampling at every third clock pulse t+2+3k.

4. (a) (i) $\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$ (ii) $\begin{bmatrix}0\\1\\1\\0\\0\\1\end{bmatrix}$

(b) (i) $\begin{bmatrix}0\\1\\0\\1\\0\\0\\0\\0\\1\\0\end{bmatrix}$ (ii) $\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}$ (iii) $\begin{bmatrix}1\\1\\0\\0\\1\\0\\0\\0\\0\\1\end{bmatrix}$

(c) (i) $\begin{bmatrix}1\\0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\1\\1\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}$ (ii) $\begin{bmatrix}0\\0\\0\\0\\0\\1\\1\\0\\1\\1\\0\\0\\1\\1\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}$ (iii) $\begin{bmatrix}1\\0\\0\\0\\1\\0\\0\\0\\0\\1\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$

5. (a)
$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & ? & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & ? & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & ? & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} (z_1 \cap \bar{z}_2) \cup (z_2 \cap \bar{z}_3) &= f_1 \\ (\bar{z}_1 \cap \bar{z}_2) \cup (z_1 \cap z_2) \cup z_3 &= f_2 \\ \bar{z}_1 \cup (z_2 \cap \bar{z}_3) &= f_3 \end{aligned}$$

(b)
$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\{\bar{z}_2\cap[(\bar{z}_1\cap z_3)\cup(z_1\cap\bar{z}_3)]\}\cup\bar{z}_4=f_1$$
$$[\bar{z}_2\cap(z_1\cup\bar{z}_4)]\cup(z_2\cap\bar{z}_3\cap z_4)=f_2$$
$$\{z_3\cap[\bar{z}_2\cup(\bar{z}_1\cap z_4)]\}\cup\{\bar{z}_4\cap[(\bar{z}_1\cap\bar{z}_2)\cup(z_2\cap\bar{z}_3)]\}=f_3$$
$$\{\bar{z}_2\cap[(\bar{z}_1\cap\bar{z}_4)\cup(z_1\cap z_4)]\}\cup\{z_3\cap[\bar{z}_1\cup(z_2\cap\bar{z}_4)]\}=f_4$$

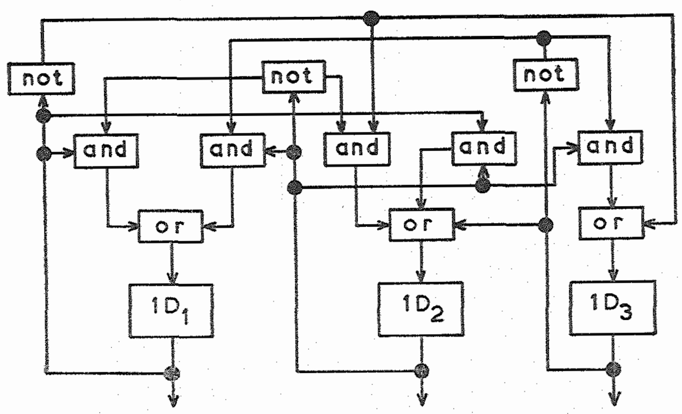
(*c*)
$$B=\begin{bmatrix}0&1&1&1&1&0&0&0\\1&1&1&0&0&0&1&0\\1&1&0&0&1&0&0&1\end{bmatrix}$$

$$(\bar{z}_1\cap\bar{z}_2\cap z_3)\cup[\bar{z}_3\cap(z_1\cup z_2)]=f_1$$
$$(\bar{z}_1\cap z_2)\cup(\bar{z}_2\cap\bar{z}_3)=f_2$$
$$(z_1\cap z_2\cap z_3)\cup[\bar{z}_2\cap(\bar{z}_1\cup\bar{z}_3)]=f_3$$

(*d*)
$$B=\begin{bmatrix}0&1&0&1&0&1&0&1\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&1&0&0\\0&0&0&0&1&1&0&1\\0&0&0&0&0&0&1&1\end{bmatrix}$$

$$z_1=f_1$$
$$0=f_2$$
$$\bar{z}_1\cap z_2=f_3$$
$$z_1\cap[(z_2\cap\bar{z}_3)\cup(\bar{z}_2\cap z_3)]=f_4$$
$$z_3\cap(z_1\cup\bar{z}_2)=f_5$$
$$z_2\cap z_3=f_6$$

6. (*a*)



(*b*)